Introductory Course Mathematics

PD Dr. Olaf Schnürer

University of St. Gallen

Fall semester 2025

Overview

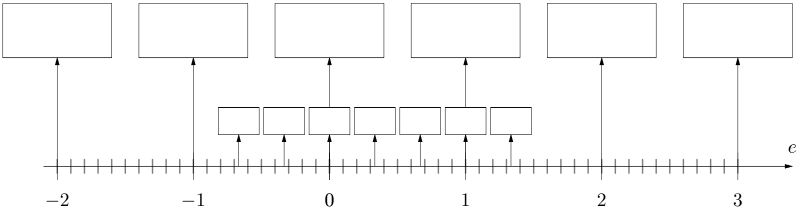
Exponentiation, definition of b^e

Question

How to define the power b^e? here $b > 0$ is the **base** and $e \in \mathbb{R}$ is the **exponent**

Note: The letter e in the present discussion has nothing to do with the *Euler number* e which will be used later.

We fix $b > 0$ and write in each box what b^e should be, depending on e on the horizontal number line. What is done here during the lecture is written down on the next slide - so no worries if I'm too fast.



Now b^e is defined for all rational numbers $e \in \mathbb{Q}$.
How to extend this to all real numbers $e \in \mathbb{R}$? Possible definition as a limit of a sequence:

$$b^\pi = b^{3.14159\ldots} := \text{limit of the sequence } b^3,\ b^{3.1},\ b^{3.14},\ b^{3.141},\ b^{3.1415},\ldots \text{ of real numbers}$$

Exponentiation, definition of b^e

Proceed as follows:

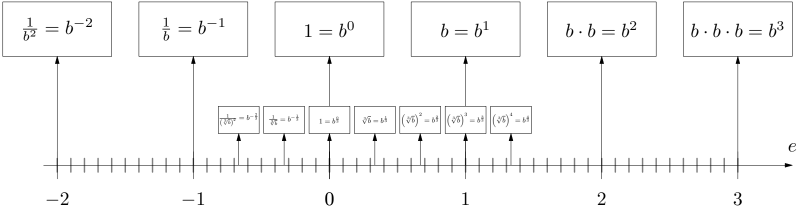
1. fill large boxes with **positive integer/natural** exponents
2. note that passing from one large box to its neighbor on the right is multiplying by b
3. passing from one large box to its neighbor on the left should be dividing by b
4. fill large box with exponent **0**
5. fill large boxes with **negative integer** exponents

Now b^e is defined for all integers $e \in \mathbb{Z}$. Up to now b might also be negative, but better not in the following.
How to extend to rational numbers $e \in \mathbb{Q}$?

6. fill small boxes above 0 and 1

We want that passing from one little box to its neighbor on the right is multiplying by some number. This number must be $\sqrt[3]{a}$. Similarly, going to the left should be dividing by this number.

7. fill small boxes (first positive, then negative exponents)



Exponentiation rules

Laws of exponents = exponent(iation) rules

For all bases $a, b \in (0, \infty)$ and all exponents $e, f \in \mathbb{R}$ we have:

$a^e \cdot a^f = a^{e+f}$	addition of exponents if **same** base
$a^e \cdot b^e = (ab)^e$	multiplication of bases if **same** exponent
$(a^e)^f = a^{ef}$	iterated exponentiation

The formal proof of these rules requires some work.
As with all equalities: Depending on the problem, it might be appropriate to use these equations "from left to right" resp. "from right to left".

Examples

From the above rules, we deduce the following equalities. Some of these consequences are also called laws of exponents.

$$a^{-e} = a^{e\cdot(-1)} = (a^e)^{-1} = \frac{1}{a^e}$$

$$= a^{-e} = a^{(-1)\cdot e} = \left(a^{-1}\right)^e = \left(\frac{1}{a}\right)^e$$

$$a^{e-f} = a^{e+(-f)} = a^e \cdot a^{-f} = a^e \cdot \frac{1}{a^f} = \frac{a^e}{a^f}$$

$$\left(\frac{a}{b}\right)^e = \left(a \cdot \frac{1}{b}\right)^e = a^e \cdot \left(\frac{1}{b}\right)^e = a^e \cdot \frac{1}{b^e} = \frac{a^e}{b^e}$$

Examples

$$\left(\sqrt[m]{a}\right)^e = a^{\frac{e}{m}} = a^{e \cdot \frac{1}{m}} = \left(a^e\right)^{\frac{1}{m}} = \left(\sqrt[m]{a^e}\right)$$

$$\sqrt[3]{x^2} \cdot \sqrt[7]{x} = x^{\frac{2}{3}} \cdot x^{\frac{1}{7}} = x^{\frac{2}{3}+\frac{1}{7}} = x^{\frac{14+3}{21}} = x^{\frac{17}{21}} = \sqrt[21]{x^{17}}$$

Power functions vs. exponential functions

When considering powers b^e, we may

1. fix the exponent and vary the base or
2. fix the base and vary the exponent.

The first case leads to **power functions**, i.e. functions of the form any $a \in \mathbb{R}$

$$f(x) = x^a$$

The second case leads to **exponential functions**, i.e. functions of the form any $a > 0$

$$f(x) = a^x$$

Power functions

Definition (power functions)

A **power function** is a function f of the following form (where $a \in \mathbb{R}$ is any real number).

$$f : (0, \infty) \to (0, \infty)$$
$$x \mapsto y = f(x) = x^a$$

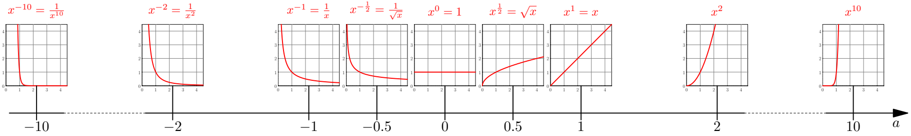
In some books, a non-zero factor is also allowed, so that the general form of a power function is $f(x) = cx^a$ for $a \in \mathbb{R}$ and $c \in \mathbb{R} \setminus \{0\}$.

Examples (Power functions)

$f(x) = x^3$ or $f(x) = x^{\frac{1}{2}} = \sqrt{x}$ or $f(x) = x^{-2} = \frac{1}{x^2}$ or $f(x) = x^{-\frac{2}{3}} = \frac{1}{\left(\sqrt[3]{x}\right)^2} = \frac{1}{\sqrt[3]{x^2}}$

Power functions

Geogebra: Graphs of x^a depending on $a \in \mathbb{R}$. Static picture here:



Properties of power functions $f(x) = x^a$ $(a \in \mathbb{R})$

- set-theoretic property:
  - bijective as a function $(0, \infty) \to (0, \infty)$ if $a \neq 0$, hence invertible (= has an inverse function); the inverse function is itself a power function, namely

$$f^{-1}(x) = x^{\frac{1}{a}}$$

Proof: $f(f^{-1}(x)) = \left(x^{\frac{1}{a}}\right)^a = x^{\frac{1}{a}\cdot a} = x^1 = x$ and $f^{-1}(f(x)) = (x^a)^{\frac{1}{a}} = x^{a\cdot\frac{1}{a}} = x^1 = x$

- growth properties:
  - strictly increasing if $a > 0$
  - strictly decreasing if $a < 0$
  - constant if $a = 0$

- curvature properties:
  - strictly convex if $a \in (-\infty, 0) \cup (1, \infty)$ "left turn"
  - strictly concave if $a \in (0, 1)$ "right turn"
  - linear if $a = 0$ or $a = 1$

Definition (growth properties: (strictly) increasing/decreasing)

A function $f\colon X \to Y$ between subsets of $\mathbb{R}$ is

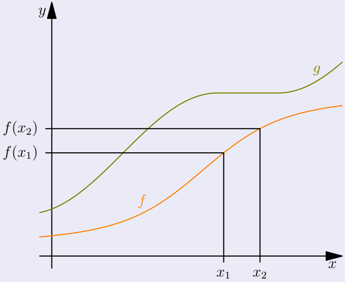
- **strictly increasing** if and only if

$$x_1 < x_2 \implies f(x_1) < f(x_2) \quad \text{for all } x_1, x_2 \in X.$$

- **strictly decreasing** if and only if

$$x_1 < x_2 \implies f(x_1) > f(x_2) \quad \text{for all } x_1, x_2 \in X.$$

For the versions without "strictly", replace $f(x_1) < f(x_2)$ by $f(x_1) \leq f(x_2)$ resp. $f(x_1) > f(x_2)$ by $f(x_1) \geq f(x_2)$.



The function f in the picture is strictly increasing. The function g is increasing but not strictly increasing.

(Look ahead to derivatives)

The first derivative $f'(x)$ knows whether f is strictly increasing or decreasing.

- $f'(x_0) > 0 \implies f$ strictly increasing near x_0.
- $f'(x_0) < 0 \implies f$ strictly decreasing near x_0.

Power functions

Example (maybe omitted in lecture)

Why is $f(x) = x^{\frac{3}{7}} = \sqrt[7]{x^3} = \left(\sqrt[7]{x}\right)^3$ strictly increasing on $(0, \infty)$?

Proof.

Step 1: We claim that $g(x) = x^3$ is strictly increasing on $(0, \infty)$.
Let $x < y$ be arbitrary elements of $(0, \infty)$. We multiply this inequality by $x > 0$ and by $y > 0$ and obtain

$$x^2 = x \cdot x < xy < y \cdot y = y^2 \qquad \text{so} \qquad x^2 < y^2$$

Multiplying this inequality $x^2 < y^2$ by $x > 0$ and the inequality $x < y$ by $y^2 > 0$ yields

$$x^3 = x^2 \cdot x < y^2 \cdot x < y^2 \cdot y = y^3 \qquad \text{so} \qquad x^3 < y^3$$

Step 2: Similarly one sees that x^7 is strictly incrasing. Hence its inverse function $\sqrt[7]{x} = x^{\frac{1}{7}}$ is strictly increasing as well.

General fact: A bijective function is strictly increasing if and only if its inverse function is strictly increasing.

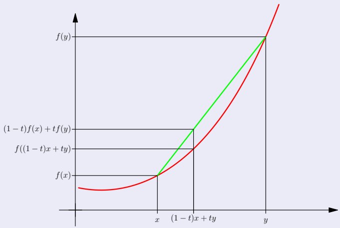
Step 3: We have $f = r \circ g$ for $r(x) = \sqrt[7]{x}$ and $g(x) = x^3$. Since both g and r are strictly increasing, the same is true for their composition $f(x) = (r \circ g)(x) = r(g(x)) = \sqrt[7]{x^3}$.

General fact: Every composition of strictly increasing functions is strictly increasing. □

Definition (curvature properties: convex, concave)

A function $f\colon I \to Y$ from an intervall I to a subset of $\mathbb{R}$ is

- **convex** if any line segment between any two points of the graph of f lies above this graph, cf. picture; left turn
- **concave** if any line segment between any two points of the graph of f lies below this graph. right turn



For the versions with "strictly", require only the "inner part of the line segments" to lie strictly above/below the graph.

Formally, convexity is defined by the following condition.

$$f\big((1-t)\cdot x + t\cdot y\big) \le (1-t)\cdot f(x) + t\cdot f(y) \qquad \text{for all } x, y \in X \text{ and } t \in [0,1].$$

Understanding $(1-t)\cdot x + t\cdot y$.

- For $t = 0$ this number is x.
- For $t = 1$ this number is y.
- We have $(1-t)\cdot x + t\cdot y = x - t\cdot x + t\cdot y = x + t(y-x)$.
  So this number is x plus t times the (1-dimensional) vector $y - x = \overrightarrow{xy}$.
- If the (time variable) t increases uniformly from 0 to 1, this number moves uniformly from x to y (on the x-axis).

Similarly, the number $(1-t)f(x) + tf(y)$ moves uniformly from $f(x)$ to $f(y)$ (on the y-axis).
The inequality in the formal definition of convexity requires that this moving point on the y-axis is always higher than the image (under f) of the moving point on the x-axis.

(Look ahead to derivatives)

The second derivative $f''(x)$ knows whether f is concave or convex.

- $f''(x_0) > 0 \implies f$ is convex near x_0.
- $f''(x_0) < 0 \implies f$ is concave near x_0.

Example

The function $f(x) = x^2$ has derivative $f'(x) = 2x$ and second derivative $f''(x) = 2$.

So $f''(x_0) = 2 > 0$ for all x_0 and f is convex everywhere (left turn), as this should be for the parabola.

Exponential functions

Definition (exponential functions)

An **exponential function** is a function f of the following form (where $a \in (0, \infty)$).

$$\begin{aligned} f: \mathbb{R} &\to (0, \infty) \\ x &\mapsto y = f(x) = a^x \end{aligned}$$

In some books, a non-zero factor is also allowed, so that the general form of an exponential function is $f(x) = ca^x$ for $a \in (0, \infty)$ and $c \in \mathbb{R} \setminus \{0\}$. Sometimes, $a = 1$ is not allowed.

Exponential functions are used to describe the growth or decay of some quantities.

$$f(x+1) = a^{x+1} = a \cdot a^x = a \cdot f(x) \qquad \text{for each } x \in \mathbb{R}$$

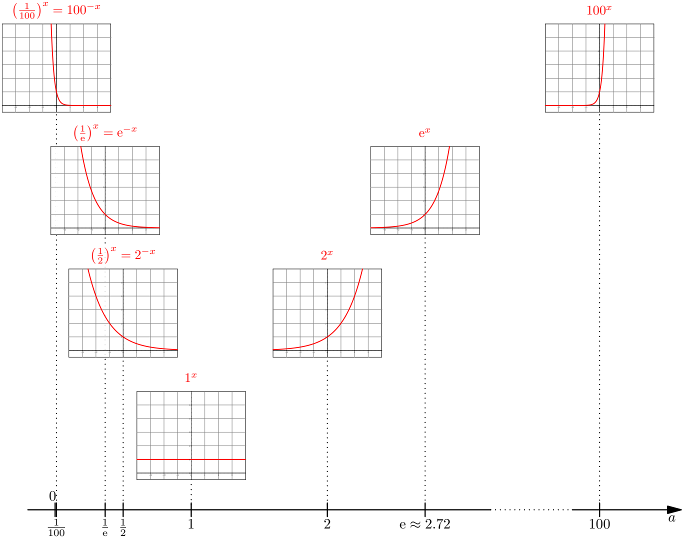
I.e.: The "quantity" $f(x)$ growths by the factor a whenever the "time" x increases by 1. For $a > 1$ we have growth, for $0 < a < 1$ decay.

Examples (Exponential functions)

- $f(x) = 3^x$ growth by factor 3 per time unit (some population of bacteria)
- $f(x) = \left(\frac{1}{2}\right)^x$ growth by factor $\frac{1}{2}$ per time unit (radioactive decay, 1 unit = half-life of radioactive element, e.g. uranium)

Exponential functions

Geogebra: Graphs of a^x depending on $a \in (0, \infty)$. Static picture here:



Properties of exponential functions $f(x) = a^x$ $(a > 0)$

- set-theoretic property:
  - bijective if $a \neq 1$, hence invertible; the inverse function is called the **logarithm to base** a and denoted as follows.

    $$f^{-1}(x) = \log_a(x)$$

    Note that

    $$\log_a \colon (0, \infty) \to \mathbb{R}$$

    I.e. logarithms are defined for **positive** arguments.
- growth properties:
  - strictly decreasing if $0 < a < 1$ — same true for $\log_a$
  - strictly increasing if $a > 1$ — same true for $\log_a$
  - constant if $a = 1$

  because inverse function has same growth property
- curvature properties:
  - strictly convex if $a \neq 1$ "left turn" — $\log_a$ strictly concave "right turn"
  - constant (and convex) if $a = 1$

  because inverse of a convex function is concave (graph is mirror image)
- algebraic property:

$$f(x+y) = a^{x+y} = a^x \cdot a^y = f(x) \cdot f(y)$$

Exponential functions are "plus-times-compatible".

Official terminology:
Exponential functions are group morphism from the additive group $(\mathbb{R}, +)$ to the multiplicative group $((0, \infty), \cdot)$.

$$\log_a(x \cdot y) = \log_a(x) + \log_a(y)$$

Logarithms are "times-plus-compatible".

Official terminology:
Logarithms are group morphism from the additive group $(\mathbb{R}, +)$ to the multiplicative group $((0, \infty), \cdot)$.

Logarithm rules = log rules

By definition, $f^{-1} = \log_a$ is inverse to $f(x) = a^x$. In particular:

$$\log_a(x) = f^{-1}(x) = \big(\text{the unique element } z \in \mathbb{R} \text{ with } a^z = x\big)$$
$$= \text{"}a \text{ raised to which number/exponent is } x\text{?"}$$

In particular, $\log_a(a^x) = x$ and $a^{\log_a(y)} = y$. for all $a > 0$, $x \in \mathbb{R}$, $y > 0$

Also: $\log_a(y) = x \iff a^x = y$. for all $a > 0$, $x \in \mathbb{R}$, $y > 0$

Log rules

For all positive real numbers $a, b \in (0, \infty)$ and $x, y \in (0, \infty)$ we have:

$$\log_a(xy) = \log_a(x) + \log_a(y) \qquad \text{logarithm of a product}$$
$$\log_a(x^y) = y \cdot \log_a(x) \qquad \text{logarithm of a power}$$

For the proof, use the exponentiation rules $a^e \cdot a^f = a^{e+f}$ and $(a^e)^f = a^{ef}$ and the fact that $\log_a(x)$ is inverse to a^x.

These rules imply:

$$\log_a\left(\frac{1}{y}\right) = -\log_a(y)$$
$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$
$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)} \qquad \text{base change}$$

Number of digits

The number of digits of a number x written as usual in the decimal system is roughly

$$\log_{10}(x)$$

Examples

- The 7-digit number $x = 1'000'000$ has approximately $\log_{10}(1'000'000) = 6$ digits.
- The 7-digit number $x = 9'999'999$ has approximately $\log_{10}(9'999'999) \approx 6.99999995657055$ digits.

The precise formula is (number of digits of x) $= \lfloor \log_{10}(x) \rfloor + 1$ where "floor y" $\lfloor y \rfloor$ is "y rounded down".

exp and ln

For mathematicians, the most important exponential function is the exponential function whose base is **Euler's number** $\mathrm{e} = 2.71828\ldots$, i.e. $f(x) = \mathrm{e}^x$. This function is often denoted by exp and usually just called **the exponential function**.

$$\exp\colon \mathbb{R} \to (0,\infty) \qquad \ln\colon (0,\infty) \to \mathbb{R}$$

$$x \mapsto \boxed{\exp(x) = \mathrm{e}^x = \ln^{-1}(x)} \qquad x \mapsto \boxed{\ln(x) = \log_{\mathrm{e}}(x) = \exp^{-1}(x)}$$

Its inverse function is the **natural logarithm** and denoted by ln ("logarithmus naturalis").

Both functions can be computed using power series: $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and $\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$. E.g. $\mathrm{e} = \mathrm{e}^1 = \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$.

Important identities

$$\exp(x+y) = \mathrm{e}^{x+y} = \mathrm{e}^x \cdot \mathrm{e}^y = \exp(x) \cdot \exp(y)$$

using new exp-notation

$$\ln(x \cdot y) = \ln(x) + \ln(y)$$

$$\ln(x^y) = y \cdot \ln(x)$$

special cases of above rules for $\ln = \log_{\mathrm{e}}$

Any power can be computed using exp and ln.

and that is how computers do compute powers

$$a^b = \exp(\ln(a^b)) = \exp(b \cdot \ln(a)) = \mathrm{e}^{b \ln(a)}$$

Any logarithm can be computed using ln.

and that is what computers do

$$\log_a(x) = \frac{\log_{\mathrm{e}}(x)}{\log_{\mathrm{e}}(a)} = \frac{\ln(x)}{\ln(a)}$$

by base change

A time-dependent quantity grows/falls exponentially if it is strictly increasing/decreasing and is multiplied by the same factor in each fixed time interval (regardless of when this time interval begins).

Quantities that grow exponentially are described by functions of the form

$$f(t) = f_0 \cdot b^{\frac{t}{T}} = f_0 \cdot \mathrm{e}^{\frac{t}{T} \cdot \ln(b)} = f_0 \cdot \mathrm{e}^{\frac{\ln(b)}{T} \cdot t} = f_0 \cdot \left(e^{\frac{\ln(b)}{T}}\right)^t$$

where f_0 is the initial value of the quantity at time $t = 0$ and b is the growth factor/growth rate during an arbitrarily chosen time period $T > 0$.

Example (from biology)

If a cell population grows exponentially by a factor of 2 every 5 hours, the number of cells at time t (measured in hours) is described by

$$f(t) = f_0 \cdot 2^{\frac{t}{5}} = f_0 \cdot \mathrm{e}^{\frac{t}{5} \cdot \ln(2)} = f_0 \cdot e^{\frac{\ln(2)}{5} \cdot t} = f_0 \cdot \left(e^{\frac{\ln(2)}{5}}\right)^t$$

where f_0 is the inital number of cells. (For testing purposes, set $t = 0$ or $t = 5$ or $t = 10$.)
From the last expression, you can see either directly (because it has the form $f_0 \cdot b^{\frac{t}{T}}$ for $T = 1$) or by one of the following calculations that the growth factor per hour is $e^{\frac{\ln(2)}{5}}$.

$$\frac{f(1)}{f(0)} = \frac{f_0 \cdot e^{\frac{\ln(2)}{5}}}{f_0} = e^{\frac{\ln(2)}{5}} \quad \text{oder} \quad \frac{f(t+1)}{f(t)} = \frac{f_0 \cdot e^{\frac{\ln(2)}{5} \cdot (t+1)}}{f_0 \cdot e^{\frac{\ln(2)}{5} \cdot t}} = \frac{f_0 \cdot e^{\frac{\ln(2)}{5} \cdot t} \cdot e^{\frac{\ln(2)}{5}}}{f_0 \cdot e^{\frac{\ln(2)}{5} \cdot t}} = e^{\frac{\ln(2)}{5}}$$

In the math lecture, the formula $A_{n,\infty} = A(n) = P \cdot \mathrm{e}^{i \cdot n} = P \cdot (\mathrm{e}^i)^n$ appears in connection with continuous compunding. There, the growth factor per year is e^i.

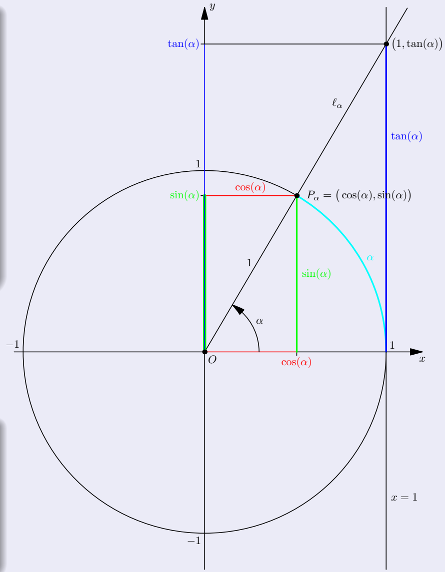
Trigonometric functions

The **unit circle** is the circle around the origin with radius 1 unit.

Definition (sin, cos, tan)

Let α be an angle. Let P_α be the point on the unit circle such that α is the angle between the positive x-axis and the ray $\ell_\alpha = OP_\alpha$. Define (cf. illustration)

- $\cos(\alpha) := \big(x\text{-coordinate of } P_\alpha\big)$
- $\sin(\alpha) := \big(y\text{-coordinate of } P_\alpha\big)$
- $\tan(\alpha) := \frac{\sin(\alpha)}{\cos(\alpha)} = \big(\text{slope of } \ell_\alpha\big)$



Note that $\tan(\alpha)$ is the y-coordinate of the intersection point of the line ℓ_α and the line $x = 1$.

(pythagorean identity)

$$(\sin(\alpha))^2 + (\cos(\alpha))^2 = 1$$

short notation: $\sin^2(\alpha) + \cos^2(\alpha) = 1$

Proof: Use Pythagoras in the obvious right-angled triangle.

At school, the trigonometric functions are often defined using right-angled triangles:

$$\sin(\alpha) = \frac{\text{opposite of } \alpha}{\text{hypotenuse}}$$
$$\cos(\alpha) = \frac{\text{adjacent of } \alpha}{\text{hypotenuse}}$$
$$\tan(\alpha) = \frac{\text{opposite of } \alpha}{\text{adjacent to } \alpha}$$

SOH = sine=opposite / hypotenuse

CAH = cosine=adjacent / hypotenuse

TOA = tangent=opposite / adjacent

Mnemonic (= trick for remembering something):

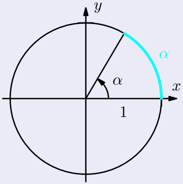
SOH-CAH-TOA

These two possible definitions are equivalent: In our definition using the unit circle, use the obvious right-angled triangle with red and green sides.
Its hypotenuse has length 1, so $\sin(\alpha) = \frac{\text{opposite green leg}}{\text{hypotenuse}} = \frac{\text{green}}{1} = \text{green}$ and similarly for $\cos(\alpha)$ and (using another right-angled triangle) $\tan(\alpha)$.

Angles are traditionally measured in degrees. A more natural[1] definition is the following.

Definition (arc measure)

The **arc measure** of an angle α is the length of the arc on the unit circle "cut out by alpha".



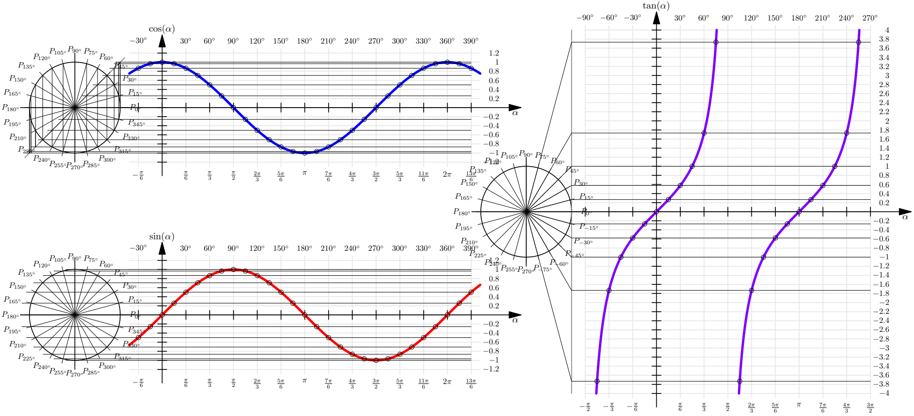
Examples

- $360^\circ = 2\pi$, $180^\circ = \pi$, $90^\circ = \frac{\pi}{2}$, $60^\circ = \frac{\pi}{3}$
- $1^\circ = \frac{\pi}{180}$, $\frac{180^\circ}{\pi} = 1$
- $540^\circ = 3\pi$
- $-180^\circ = -\pi$

The SI unit of the arc measure is radians, abbreviated rad, e.g. $180^\circ = \pi\,\text{rad}$. It is nearly always omitted in mathematics.

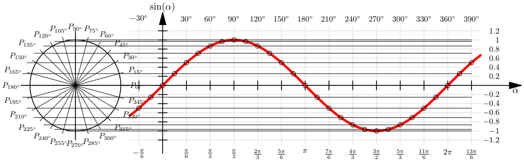
[1] Why should the full angle be 360° and not 5040 or 1 or 42? The arc measure can also be defined more naturally using any circle instead of the unit circle: divide the length of the arc, that is cut out by the angle, by the radius.

The trigonometric functions cos and sin are by definition the coordinates of a point P_α that moves with constant speed (1 unit per time unit) on the unit circle in anticlockwise direction, starting at $(1, 0)$. The variable α is both the time and the angle (in arc measure).



sine is a shift of cosine (and conversely) $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$

Properties of the sine function



- The image/range of the sine function is $[-1, 1]$.

$$\sin\colon \mathbb{R} \to [-1, 1]$$

- The zeros of the sine function are precisely the multiples of π.

$$\sin(x) = 0 \quad \iff \quad x = n\pi \text{ for some } n \in \mathbb{Z}$$

- The sine function is periodic with "period length" equal to 2π.

$$\sin(x) = \sin(x + 2\pi)$$

- The sine function is point symmetric with respect to the origin. (and w.r.t. any of its zeros)

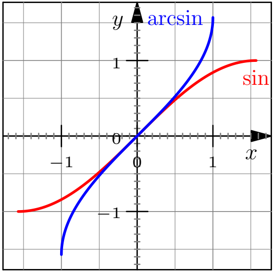
$$\sin(-x) = -\sin(x)$$

- If we shrink the source of sin to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we obtain a bijective (strongly increasing) function

$$\sin\colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$$

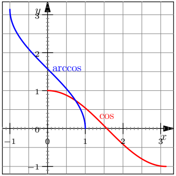
  Its inverse function is the **arcsine** function

$$\arcsin\colon [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = [-90^\circ, 90^\circ]$$

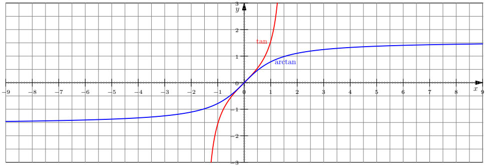


Similarly, one may discuss the properties of cos and tan. Let us at least mention their inverse functions.

- The cosine function cos is invertible as a function $\cos\colon [0, \pi] \to [-1, 1]$. Its inverse is the **arccos** function arccos.



- The tangent function tan is invertible as a function $\cos\colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$. Its inverse is the **arctan** function arctan.



Trigonometric identities

There are many relations between the trigonometric functions.
For example one can solve the pythagorean identity for $\cos(x)$ or $\sin(x)$ and obtains

$$\sin(x) = \pm\sqrt{1 - \cos^2(x)}$$
$$\cos(x) = \pm\sqrt{1 - \sin^2(x)}$$

Another example are the **angle sum identities**:

$$\sin(\alpha + \beta) = \sin(\alpha) \cdot \cos(\beta) + \cos(\alpha) \cdot \sin(\beta)$$
$$\cos(\alpha + \beta) = \cos(\alpha) \cdot \cos(\beta) - \sin(\alpha) \cdot \sin(\beta)$$

These angle sum identities are actually easy to obtain as soon as one knows complex numbers and
Euler's formula:

$$e^{ix} = \cos(x) + i \sin(x)$$

For a pretty long list of identities, see
https://en.wikipedia.org/wiki/List_of_trigonometric_identities.

Harmonic oscillation (omitted in lecture)

In order to simulate many periodic processes, one may use the following function.

$$y(t) = y_0 \cdot \sin(2\pi \cdot f \cdot t + \varphi_0)$$

In physics, this gives the position of a weight attached to a linear spring. Here

- y_0 is the amplitude;
- f is the frequency (number of repetitions per time unit); $T = \frac{1}{f}$ is the period.
- φ_0 is the phase shift (determines the starting position at time $t = 0$).

The picture shows the graph of $y(t) = 2 \cdot \sin(2\pi \cdot \frac{1}{3} \cdot t + \frac{\pi}{3})$