Introductory Course Mathematics

PD Dr. Olaf Schnürer

University of St. Gallen

Fall semester 2025

Topics during the semester

1. Basics on functions of one real variable I: Functions, absolute value, inequalities, summation sign
2. Basics on functions of one real variable I: Power functions, exponential and logarithmic functions, trigonometric functions
3. Basics on functions of one real variable II: Rules of differentiation
4. Basics on functions of one real variable II: Derivatives and properties of functions (monotonicity, extrema)
5. Partial derivatives (used in the course on macroeconomics)
6. Algebra: Linear, quadratic and exponential equations

Break

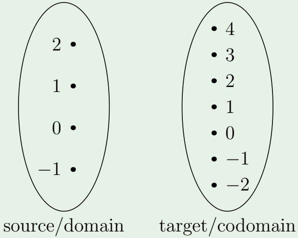
7. Analytical geometry: Visualization of functions in two variables, contour lines, general concepts
8. Analytical geometry: Contour lines, curves of second order (ellipse, parabola, hyperbola)
9. Optimization of functions of two real variables under constraints
10. Systems of equations (linear and non-linear), systems of inequalities
11. Combinatorics – the art of counting
12. Basics of linear algebra (coordinate systems, vectors, distances and angles, lines and planes)

Overview

- Functions
- Absolute value
- Inequalities
- Summation sign $\sum$

Functions

Example (of a function)



Consider two finite sets: $X = \{-1, 0, 1, 2\}$ and
$Y = \{-2, -1, 0, 1, 2, 3, 4\}$

Notation:

$$f : X \to Y \qquad \text{read: "}f\text{ from }X\text{ to }Y\text{"}$$

$$x \mapsto f(x) = x^2 \qquad \text{"}x\text{ is mapped to }f(x) = x^2\text{"}$$

Short notation: $f(x) = x^2$

Definition (function = mapping)

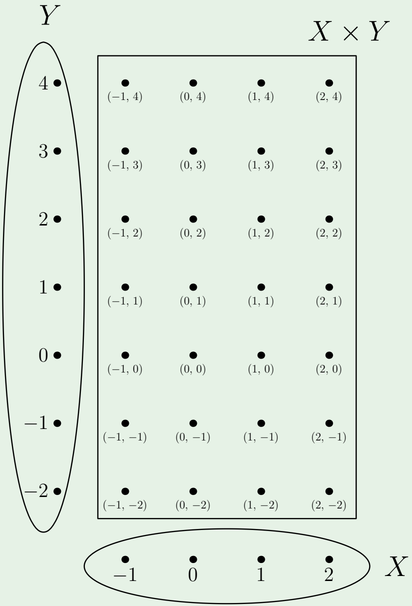
A **function** or **mapping** f from a set X to a set Y is a rule that assigns

- to *each* element x of the set X
- *exactly one* element $f(x)$ of the set Y.

Terminology:

- X is the **domain** or **source** of f;
- Y is the **codomain** or **target** of f;
- $f(x)$ is the **image/value of** x **under** f;
- $f(X) = \{f(x) \mid x \in X\}$ is the **range** or **image** of f.

Example (graph of a function)

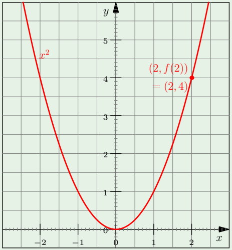


For every element $x \in X = \{-1, 0, 1, 2\}$, mark the point $(x, f(x)) = (x, x^2)$ in the product $X \times Y$. The set of points obtained in this way is the **graph** of f. It is our main tool to visualize functions.

Let's pass from finite sets to infinite sets. We still consider the "squaring function" but change domain and target. We take the set $\mathbb{R}$ of all reals numbers as domain and target, i. e. we consider the following function:

$$f \colon \mathbb{R} \to \mathbb{R}$$
$$x \mapsto y = f(x) = x^2$$

Its graph is the parabola $y = x^2$.



$$\text{graph}(f) = \{(x, x^2) \mid x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$$

Definition (composition of functions)

Assume that f and g are functions such that the target of f is equal to (or contained in) the source of g:

$$X \xrightarrow{f} Y \xrightarrow{g} Z, \qquad X \xrightarrow{g \circ f} Z$$

Then we define a new function from X to Z, called the **composition** of g and f and denoted by $g \circ f$ (read "g after f"), by

$$(g \circ f)(x) = g(f(x)) \qquad \text{for arbitrary } x \in X$$

Note that the function g is applied *after* applying f (even though g comes before f in the expression $g \circ f$).

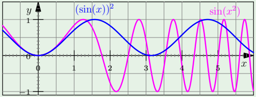
Example

The function $h(x) = \sin(x^2)$ is the composition of the sine function $g(x) = \sin(x)$ and the squaring function $f(x) = x^2$ because $(g \circ f)(x) = g(f(x)) = g(x^2) = \sin(x^2) = h(x)$.

$$\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g=\sin} \mathbb{R}, \qquad \mathbb{R} \xrightarrow{h = g\circ f = \sin \circ f} \mathbb{R}$$

Note that the order matters: The composition $f \circ g$ is given by $f(g(x)) = f(\sin(x)) = (\sin(x))^2$ which is not equal to $h(x) = \sin(x^2)$.

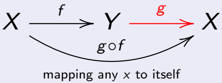
Here, $\mathbb{R}$ is source and target of f and g, so $g \circ f$ makes sense. In general, $f \circ g$ may not even be defined.



Inverse function

Motivation

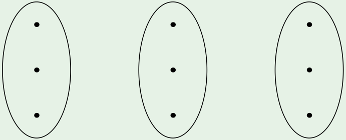
Let a function $f\colon X \to Y$ be given.
Question: Can we go back and compute x from $y = f(x)$?



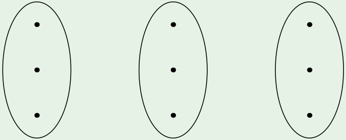
More precisely: Is there a function $g\colon Y \to X$ reversing/undoing/inverting f in the sense that the composition $g \circ f$ maps each element of X to itself, i. e. $(g \circ f)(x) = g(f(x)) = x$ for all $x \in X$?

Example (with finite sets)

Bad setting (g does not exist):



Good setting (g does exist):



- A necessary condition for the existence of g is that different elements of X are mapped to different elements of Y. A function with this property is called **injective**.
- Moreover, it would be nice if any element $y \in Y$ would be in the image of f, so that there is at least one natural candidate for $g(y)$. A function with this property is called **surjective**.

Important properties of functions

Definition (injective, surjective, bijective functions)

A function $f\colon X \to Y$ is

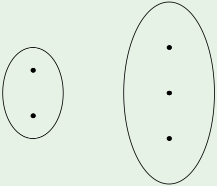
- **injective** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all elements $x_1, x_2 \in X$.
  Equivalently: $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$ for all elements $x_1, x_2 \in X$.
  In words: Distinct elements of the domain are mapped to distinct elements of the target. Each element of the target is the image of at most one element of the domain.
- **surjective** if for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$.
  In words: Every element of the target is the image of an element of the domain.
- **bijective** if it is surjective and injective.
  In words: For each element $y \in Y$ there is a **unique** element $x \in X$ with $f(x) = y$.

Examples with finite sets

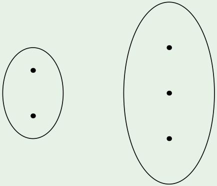
Examples

injective function:

$x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$



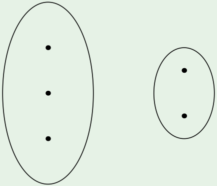
non-injective function:



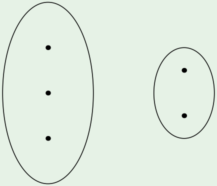
Examples

surjective function:

for any $y \in Y$ there is an $x \in X$ with $f(x) = y$



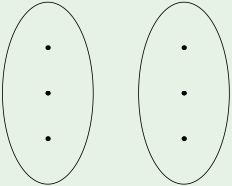
non-surjective function:



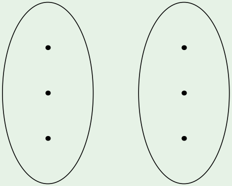
Examples

bijective function:

for any $y \in Y$ there is a unique $x \in X$ with $f(x) = y$



non-bijective function:



Example

Consider some people attending a theater performance.
Let X be the set of people and Y be the set of seats in the theater.
Let $f\colon X \to Y$ be the function/mapping that maps a person to its seat.

- f injective means: The audience is happy: No two people sit on the same seat.
- f surjective means: The theater manager is happy: The performance is sold out ... but there might be several seats being occupied by more than one person.
- f bijective means: Audience and theater manager are happy.

Let $f: X \to Y$ be a function between subsets of $\mathbb{R}$. Then

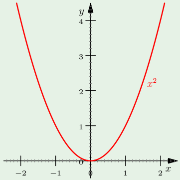
- f is injective if and only if every horizontal line through Y hits the graph at most once.
- f is surjective if and only if every horizontal line through a point of Y hits the graph.
- f is bijective if and only if every horizontal line through a point of Y hits the graph precisely once.

Example

The function

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto y = f(x) = x^2$$

- is not injective: $f(-1) = f(1)$. more generally, $f(-a) = f(a)$ for any $a \neq 0$
- is not surjective: $-1 \in \mathbb{R}$ is not in the image; no negative number in image



Replacing the source by $[0, \infty)$, we obtain an injective function

$$f: [0, \infty) \to \mathbb{R}$$
$$x \mapsto x^2$$

Replacing the target by the image $f(\mathbb{R}) = [0, \infty)$, we obtain a surjective function

$$f: \mathbb{R} \to [0, \infty)$$
$$x \mapsto x^2$$

For any function $f: X \to Y$ between arbitrary sets, replacing the target Y by the image $f(X)$ always yields a *surjective* function $f: X \to f(X)$.

Doing both, we obtain a bijective function

$$f: [0, \infty) \to [0, \infty)$$
$$x \mapsto x^2$$

Theorem (inverse function)

Let $f\colon X \to Y$ be a **bijective** *function. Then it makes sense to define a function $f^{-1}\colon Y \to X$ by*

$$f^{-1}(y) = \big(\textit{the unique element } x \in X \textit{ with } f(x) = y\big)$$

This function has the following two properties:

- $f^{-1}(f(x)) = x$ *for all* $x \in X$ — *Proof:* $f^{-1}(f(x)) = \big(\textit{the unique element } x' \in X \textit{ with } f(x') = f(x)\big) = x$
- $f(f^{-1}(y)) = y$ *for all* $y \in Y$ — *Proof:* $f(f^{-1}(y)) = f\big(\textit{the unique element } x \in X \textit{ with } f(x) = y\big) = x$

In words: f^{-1} reverts/inverts f and vice-versa.
The function f^{-1} is called the **inverse function** *or the* **inverse** *of f.*

Note: $\boxed{f(x) = y \iff x = f^{-1}(y)}$ for all $x \in X$ and $y \in Y$.

Examples

$f(x) = x + 3$	$f^{-1}(y) = y - 3$	why? $y = x + 3 \iff x = y - 3$
$f(x) = 3x$	$f^{-1}(y) = \frac{1}{3}y$	
$f(x) = x^2$	$f^{-1}(y) = \sqrt{y}$	as functions $[0, \infty) \xrightarrow{f} [0, \infty) \xrightarrow{f^{-1}} [0, \infty)$
$f(x) = x^3$	$f^{-1}(y) = \sqrt[3]{y}$	as functions $[0, \infty) \xrightarrow{f} [0, \infty) \xrightarrow{f^{-1}} [0, \infty)$
$f(x) = \mathrm{e}^x = \exp(x)$	$f^{-1}(y) = \ln(y)$	as functions $\mathbb{R} \xrightarrow{f} (0, \infty) \xrightarrow{f^{-1}} \mathbb{R}$

Example (standard method for finding the inverse function)

For many functions given by algebraic formulas there is a standard way to find the inverse function (if it exists).
Consider for example the function $f(x) = \frac{x+3}{x-2}$. Then, by definition,

$$f^{-1}(y) = \big(\text{the }_{\text{hopefully}}\text{ unique element } x \in X \text{ with } f(x) = y\big)$$

This means that we should try to solve the following equation for x.

$$\begin{aligned} f(x) = \frac{x+3}{x-2} &= y \\ x+3 &= (x-2)y \\ x+3 &= xy-2y \\ x-xy &= -2y-3 \\ x(1-y) &= -2y-3 \\ x &= \frac{-2y-3}{1-y} = \frac{2y+3}{y-1} \end{aligned}$$

All these equations are equivalent as long as $x \neq 2$ and $y \neq 1$.

This computation shows that for any $y \in \mathbb{R} \setminus \{1\}$, there is precisely one $x \in \mathbb{R} \setminus \{2\}$ satisfying $f(x) = y$.
Therefore, our original function

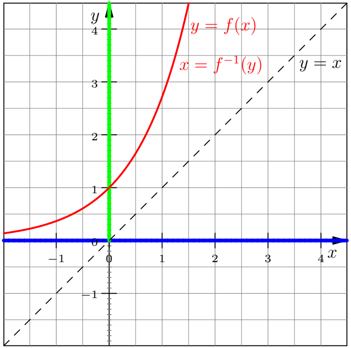
$$f\colon \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{1\}, \quad f(x) = \frac{x+3}{x-2}$$

is bijective with inverse function

$$f^{-1}\colon \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{2\}, \quad f^{-1}(y) = \frac{2y+3}{y-1}.$$

Note

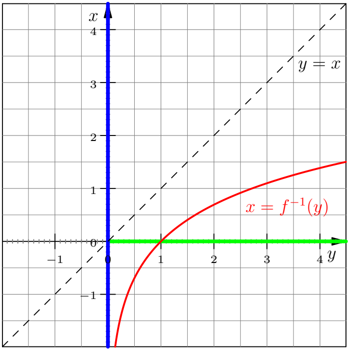
Let $f\colon X \to Y$ be a **bijective** function between subsets of $\mathbb{R}$.
Then the graph of f^{-1} is the reflection of the graph of f across the line $x = y$.



As an example, we consider the bijective exponential function

$$f\colon \mathbb{R} \to (0, \infty)$$
$$x \mapsto y = f(x) = e^x$$

The red curve is the graph of f.
The red curve is also the graph of f^{-1} if we view y as the independent variable.
Since we are used to depict the independent variable horizontally, we reflect the whole picture (including domain and target) across the line $x = y$.
The result is on the next slide.



The red curve is the graph of the inverse function

$$f^{-1}\colon (0,\infty) \to \mathbb{R}$$
$$y \mapsto x = f^{-1}(y) = \ln(y)$$

It is very common to use x as the independent variable and y as the dependent variable. This is achieved by swapping x and y. The inverse function is then denoted as follows.

$$f^{-1}\colon (0,\infty) \to \mathbb{R}$$
$$x \mapsto y = f^{-1}(x) = \ln(x)$$

To be consistent with the picture, all x and y there must be swapped as well.

Definition (absolute value)

The **absolute value** $|x|$ of a real number $x \in \mathbb{R}$ is its distance from the origin on the number line.

The formal definition is

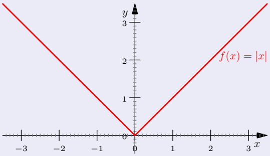
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}$$

Examples

$$|5| = 5$$
$$|-7| = -(-7) = 7$$
$$|0| = 0$$

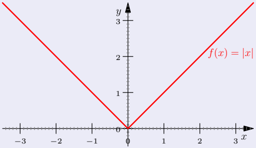
Omit the minus sign if x is negative!

The graph of the absolute value function $f(x) = |x|$.



Properties of the absolute value

- "bounded from below", $|x| \geq 0$: The absolute value is always non-negative (= never negative).
- "symmetric with respect to the y-axis": $|x| = |-x|$.



- growth properties:
  - On the "negative x-axis including zero" (= the interval $(-\infty, 0]$), the absolute value function is strictly decreasing.
  - On the "positive x-axis including zero" (= the interval $[0, \infty)$), the absolute value function is strictly increasing.

For all $x, y \in \mathbb{R}$ and $a \geq 0$ we have:

- The condition $|x| \leq a$ is equivalent to $-a \leq x \leq a$.

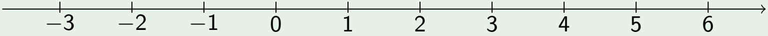
- The condition $a \leq |x|$ is equivalent to $x \leq -a$ or $a \leq x$.
- $|x \cdot y| = |x| \cdot |y|$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ $y \neq 0$
- $\sqrt{x^2} = |x|$

Note

Geometric point of view: $|a - b|$ is the distance between a and b.

Example

Solve $|x - 2| < 3$.
Geometric point of view: The distance $|x - 2|$ between x and 2 should be smaller than 3, i. e. x should be in the interval $\mathbb{L} = (2 - 3, 2 + 3) = (-1, 5)$.



Formal way of solving: $|x - 2| < 3$ is equivalent to $-3 < x - 2 < 3$, hence $-1 < x < 5$.

Example

Solve $|x - 5| = |x + 3|$. Geometric point of view:

- $|x - 5|$ is the distance between x and 5
- $|x + 3| = |x - (-3)|$ is the distance between x and -3

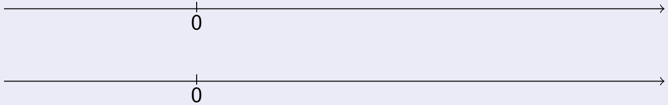
These two distances agree if and only if x is the midpoint of 5 and -3, i. e. $x = \frac{5+(-3)}{2} = 1$.

Properties of the absolute value: Triangle inequality

(Triangle inequality)

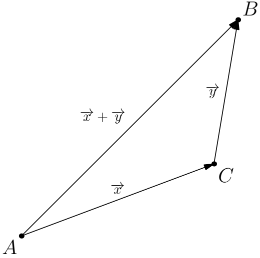
For all $x, y \in \mathbb{R}$ the following **triangle inequality** holds:

$$|x + y| \leq |x| + |y|$$



The name "triangle inequality" comes from geometry. Given any three points A, B, C in the plane, the direct route from A to B is shorter than or equal to the route via C. Using vectors as indicated in the picture, this means that

$$|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$$



In other words: In any triangle, any side is shorter than or equal to the sum of the other two sides.
In the one-dimensional setting considered above, all triangles are "degenerate" (= all points lie on a line).

Solving inequalities

When solving inequalities, the following steps do not change the solution set:

- adding/subtracting the same number/term on both sides of the inequality, e. g.

$$\begin{aligned} & x+21 < -2x+6 && \mid -x-6 \\ \Longleftrightarrow\quad & 15 < -3x \end{aligned}$$

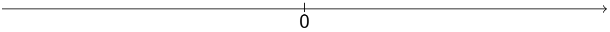
- multiplying/dividing both sides with/by a **positive** number/term, e. g.

$$\begin{aligned} & 15 < -3x && \mid \cdot \frac{1}{3}, \text{ same as } : 3 \\ \Longleftrightarrow\quad & 5 < -x \end{aligned}$$

- multiplying/dividing both sides with/by a **negative** number/term **and changing the orientation of the comparison sign**, e. g.

$$\begin{aligned} & 5 < -x && \mid \cdot (-1) \\ \Longleftrightarrow\quad & -5 > x \end{aligned}$$

  Why? Multiplying by -1 is the **order-reversing** reflection of the number line in its origin (example: from $-2 < 3 < 4$ obtain $2 > -3 > -4$).



- clearly: swapping both sides and changing the orientation of the comparison sign

$$\begin{aligned} & -5 > x \\ \Longleftrightarrow\quad & x < -5 && \text{solution set is } \mathbb{L} = (-\infty, -5) \end{aligned}$$

Example

Solve $\frac{2x+7}{x+2} \geq 1$. When multiplying by $x+2$ you need to distinguish two cases!

- Case 1, condition $x+2>0$:

$$\begin{aligned} & 2x+7 \geq x+2 && \mid -x-7 \\ \iff \quad & x \geq -5 \end{aligned}$$

  Naively, one would think that the solution set is $\mathbb{L}_1 = [-5,\infty)$. But in case 1 we assume that $x+2>0$ or equivalently $x>-2$. Hence the solution set is $\mathbb{L}_1 = (-2,\infty) \cap [-5,\infty) = (-2,\infty)$.

- Case 2, condition $x+2<0$:

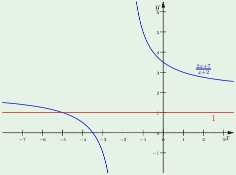
$$\begin{aligned} & 2x+7 \boxed{\leq} x+2 && \mid -x-7 \\ \iff \quad & x \leq -5 \end{aligned}$$

  Now both $x<-2$ and $x \leq -5$ must be satisfied, hence $\mathbb{L}_2 = (-\infty,-5]$.

Result: The solution set of our inequality is

$$\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2 = (-2,\infty) \cup (-\infty,-5] = (-\infty,-5] \cup (-2,\infty)$$

This is precisely the set where the blue graph in the picture is above the red line.



Definition (Summation sign = Sigma notation = $\sum$ notation)

Mathematicians use the symbol $\sum$, the capital greek letter sigma, in order to write sums of similar terms compactly. This is defined as

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

read: "sum of a_i for i from m to n"

where

- i is the **index of summation**
- a_i is a term depending on i
- the integer m is the **lower bound of summation**
- the integer n is the **upper bound of summation**

The "$i = m$" under the symbol means that the index starts out equal to m. It is then incremented by one for each summand a_i, stopping when $i = n$. Here we assume that $m \leq n$.

Example

$$\sum_{i=2}^{5} i^2 = 2^2 + 3^2 + 4^2 + 5^2$$

Examples

Sum of all natural numbers from 1 to 100: (this sum is $\frac{100 \cdot 101}{2}$)

$$\sum_{i=1}^{100} i = 1 + 2 + 3 + \cdots + 99 + 100$$

Sum of all even numbers from -4 to 100: (any variable can be the index of summation)

$$\sum_{j=-2}^{50} 2j = (-4) + (-2) + 0 + 2 + 4 + \cdots + 96 + 98 + 100$$

Sum of all odd numbers from 1 to 101:

$$\sum_{k=0}^{50} (2k + 1) = 1 + 3 + 5 + \cdots + 99 + 101 = \sum_{k=1}^{51} (2k - 1)$$

Brackets are important: (the operator $\sum$ is performed before $+$, but after $\cdot$)

$$\sum_{k=0}^{50} 2k + 1 = \left(\sum_{k=0}^{50} 2k\right) + 1 = \Big(0 + 2 + \cdots + 98 + 100\Big) + 1$$

Examples

The same sum can be written in many different forms:

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{99}{100}=\sum_{i=1}^{99}\frac{i}{i+1}=\sum_{i=2}^{100}\frac{i-1}{i}=\sum_{i=20}^{118}\frac{i-19}{i-18}$$

Index shifting: In order to see formally that the last $\sum$-expression coincides with the first one, we substitute $i=j+19$ (index shift by 19):

$$\sum_{i=20}^{118}\frac{i-19}{i-18} \overset{\substack{\text{substitution}\\ i=j+19}}{=} \sum_{j+19=20}^{118}\frac{(j+19)-19}{(j+19)-18} \qquad \text{(not official notation)}$$

$$\overset{\substack{\text{subtract 19}\\ \text{in summation}\\ \text{bounds}}}{=} \sum_{j=1}^{99}\frac{j}{j+1}$$

$$\overset{\text{rename } j \text{ to } i}{=} \sum_{i=1}^{99}\frac{i}{i+1}$$

Abstractly, the shift/substitution $i=j+s$ is given by the formula

$$\sum_{i=m}^{n}a_i=\sum_{j=m-s}^{n-s}a_{j+s}$$