

6 Functions = maps = mappings

6.0.1. The notion of a function is one of the most important notions in mathematics.

A function describes how a certain quantity depends on another quantity. For example, the position of the earth is a function of time; the «grade function» calculates the grade in an exam from the number of points of a student.

6.1 Definition of a function and first examples

Definition 6.1.1 function = map = mapping

A **function** or **map** or **mapping** is an assignement that

- assigns to *each* element of a given set, known as the **domain** (“starting set”; Definitionsmenge = Definitionsbereich, Startmenge),
- *precisely one* element of another set, known as the **codomain** (“target set”; Zielmenge).

Strictly speaking, domain and codomain must be specified for each function.
The elements of the domain are the possible **arguments** of the function.

6.1.2. The words “function” and “map” and “mapping” have the same meaning in mathematics. We will mainly use the word “function” since this is the standard word for functions whose codomain consists of real numbers.

Example 6.1.3.

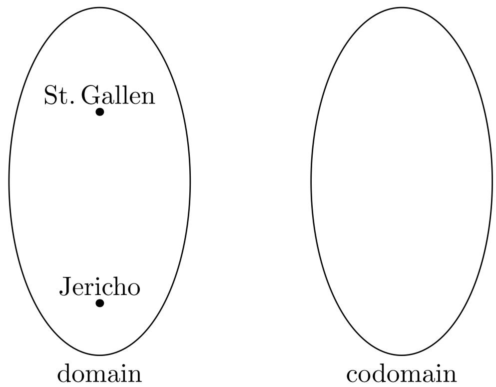
The assignement h that maps each city to its altitude above sea level is a function.
The domain of this function is

✎X = the set of all cities.

As codomain of this function we use

✎$Y = \mathbb{R}$ = the set of all real numbers

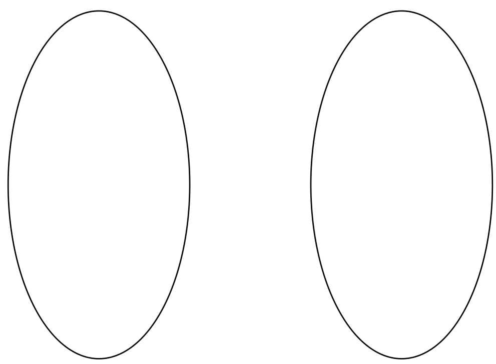
We could also use the set $\mathbb{Z}$ of all integers as the codomain if it is agreed that heights are rounded to whole meters.
The set $\mathbb{N}$ of all natural numbers would not be allowed as codomain, because there are cities that lie below sea level, such as Jericho.



Example 6.1.4.

Every table of values can be regarded as a function, for example the following table of temperatures.

Time of day in hours	8	10	12	14	16	18
Temperature in °C	-2	1	4	6	5	4



The domain is ✎

the set of all numbers in the first row, i. e. $X = \{8, 10, 12, 14, 16, 18\}$.

As codomain we can use ✎

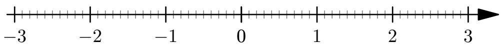
the set of all numbers occuring in the second row, i. e. $Y = \{-2, 1, 4, 5, 6\}$.

Any overset of this set could also be taken as the codomain.

Maybe ask students for further examples.

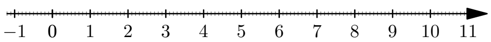
Example 6.1.5.

Let q be the square function that assigns to a number x its square $q(x) = x^2$. We can take the set $\mathbb{R}$ of real numbers as both domain and codomain of this function.



To get an impression of this function, we calculate it for a few **arguments** (= elements of the domain).

x	-3	-2	-1	0	1	2	3
$q(x) = x^2$							



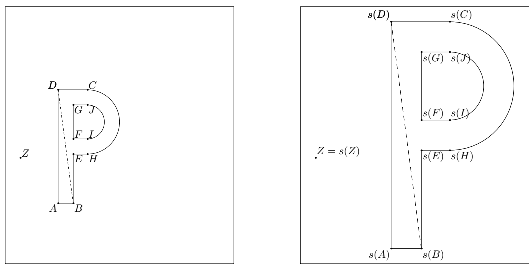
This function is explicitly written as follows: ✎

$$q\colon \mathbb{R} \to \mathbb{R}$$ "q from (the domain) $\mathbb{R}$ to (the codomain) $\mathbb{R}$"

$$x \mapsto q(x) = x^2$$ "x is mapped to q of x equal x squared"

Example 6.1.6.

The central dilation $s = s_{Z,2}$ with center Z and scale factor $\lambda = 2$ (zentrische Streckung mit Zentrum und Streckfaktor) is a function/map from the drawing plane $\mathbb{R}^2$ (= the domain) to the drawing plane $\mathbb{R}^2$ (= der codomain).
Note that domain and codomain of this map are the same set. The function goes from $\mathbb{R}^2$ to $\mathbb{R}^2$.
To each point $Q \in \mathbb{R}^2$ (= the drawing plane shown on the left) is assigned a certain point $s(Q) \in \mathbb{R}^2$ (= the drawing plane shown on the right).



6.1.7. A function can be intuitively understood as a kind of "machine" or "box" that generates an "output" (in the codomain) from any "input" (in the domain). For example, the square function q explained above can be thought of as a machine that takes any real number x as input and outputs the square x^2 of that number.

domain $\mathbb{R} \ni$ input ⟶ [squaring machine q] ⟶ Output $\in \mathbb{R}$ codomain

Similarly, one can view the central dilation function $s = s_{Z,2}$: It accepts any point Q in the plane $\mathbb{R}^2$ as input and delivers the point $s(Q)$ as output.

6.2 Notation for subsets of the real number line

6.2.1. We remind the reader of the notion of an interval and introduce some notations for sets that occur frequently when dealing with functions.

6.2.2 (Reminder)**. Intervals** (Intervalle) are connected subsets/segments of the real number line $\mathbb{R}$. For $a < b$ we use the following notation.

bounded intervals:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$
$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$
$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$
$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

unbounded intervals

$$(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$$
$$[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$$
$$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$$
$$(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$$
$$(-\infty, \infty) = \mathbb{R}$$

Definition 6.2.3

$\mathbb{R}^*$	the set of all real numbers without zero:	$\mathbb{R}^* =$ ✎$\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty) = \{x \in \mathbb{R} \mid x \neq 0\}$
$\mathbb{R}^+$	the set of all positive real numbers:	$\mathbb{R}^+ =$ ✎$(0, \infty) = \{x \in \mathbb{R} \mid x > 0\}$
$\mathbb{R}_0^+$	the set of all non-negative real numbers (i. e. positive or zero):	$\mathbb{R}_0^+ =$ ✎$[0, \infty) = \{x \in \mathbb{R} \mid x \geq 0\}$
$\mathbb{R}^-$	the set of all negative real numbers:	$\mathbb{R}^- =$ ✎$(-\infty, 0) = \{x \in \mathbb{R} \mid x < 0\}$
$\mathbb{R}_0^-$	the set of all non-positive real numbers (i. e. negative or zero):	$\mathbb{R}_0^- =$ ✎$(-\infty, 0] = \{x \in \mathbb{R} \mid x \leq 0\}$

The sets $\mathbb{Q}^*$, $\mathbb{Q}^+$, $\mathbb{Q}_0^+$, $\mathbb{Q}^-$, $\mathbb{Q}_0^-$ are defined in the same way, and analogously for $\mathbb{Z}$. For natural numbers, only the notation $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ is actually used.

Exercise E1 True or false? If the statement is false, correct it with as few changes as possible.

a) $\mathbb{R}^- = \mathbb{R} \setminus \mathbb{R}^+$ b) $\mathbb{R}^* = \mathbb{R}^+ \cup \mathbb{R}^-$ c) $\mathbb{N}^+ = \mathbb{Z}^+$

d) $\mathbb{Q}^+ \cap \mathbb{Z} = \mathbb{N}$ e) $\mathbb{R}_0^- \cap \mathbb{Q}_0^+ = \{0\}$ f) $\mathbb{R}_0^+ \cup \mathbb{Z}^- = \mathbb{Q}$

6.3 Notation of functions

Definition 6.3.1 Detailed notation of a function, argument, value at an argument

The detailed mathematical notation for a function is as follows: ✎

$$\begin{aligned} f\colon\ & X && \to && Y \\ & x && \mapsto && \text{(a rule/expression how to calculate } f(x) \text{ from } x) \end{aligned}$$

where

- f is the **name of the function** (usually any lowercase letter)
- the set X is the **domain** (= **starting set**, **source set**)
- the set Y is the **codomain** (= **target set**)
- x is the **argument**, where x stands for an arbitrary element of X. The choice of the letter x is arbitrary (but it must not be already used).
- $f(x)$ is the **value of f at x**. Example: $f(2)$ is the value of (the function) f at the point/argument 2.

The specified term for calculating $f(x)$ must be an element of the codomain Y, for any element $x \in X$ of the domain.

How to read the above notation:

$f\colon X \to Y$	Read: ✎f from X to Y.
$x \mapsto \ldots$	Read: ✎x is mapped to ....

Examples 6.3.2.

$$\begin{array}{lllll} a: \mathbb{R} \to \mathbb{R} & m: \mathbb{R} \to \mathbb{R} & k: \mathbb{R}^* \to \mathbb{R}^* & q: \mathbb{R} \to \mathbb{R}_0^+ & w: \mathbb{R}_0^+ \to \mathbb{R}_0^+ \\ \quad x \mapsto x+5 & \quad r \mapsto 3r & \quad t \mapsto \frac{1}{t} & \quad v \mapsto v^2 & \quad x \mapsto \sqrt{x} \end{array}$$

In our last three examples of functions, the codomain could have been chosen to be larger (e. g. the entire set $\mathbb{R}$), but not smaller.

All our functions go from a subset of $\mathbb{R}$ to a subset of $\mathbb{R}$. Such functions are often specified without specifying the domain and codomain as follows.

$$a(x) = x + 5 \qquad m(r) = 3r \qquad k(t) = \frac{1}{t} \qquad q(v) = v^2 \qquad w(x) = \sqrt{x}$$

The domain is then usually assumed to be the largest possible “meaningful” subset of $\mathbb{R}$ (i. e. if possible, the whole set $\mathbb{R}$), and the codomain is usually assumed to be $\mathbb{R}$.

Exercise E2 Consider the functions from example 6.3.2 and calculate the following function values.

a) $a(2)$
b) the value of m at 7
c) the value of k at 5
d) the value of q at -1
e) $a\left(\frac{2}{3}\right)$
f) $m\left(-\frac{3}{4}\right)$
g) $k\left(\sqrt{2}\right)$
h) $a(x-2)$
i) $m(u+v)$
j) $q(c+d)$
k) $a(m(4))$
l) $m(a(4))$ Is it true that $a(m(4)) = m(a(4))$?
m) $q(m(k(-0.5)))$
n) $a(k(q(\sqrt{7})))$

Exercise E3 Transform the following descriptions of functions into the mathematical notation explained in note box 6.3.1.

Hopefully, you will see that the mathematical notation is easier to read and more compact than the text.

(a) The name of the function is w, its domain is the set of all non-negative real numbers, its codomain is the set of all real numbers, and w calculates the third root of its argument.
(b) The function d goes from the set of all natural numbers to the set of all integers and multiplies each argument by -2.
(c) The function n hat the closed interval from 0 to 20 as its starting set and the closed interval from 1 to 6 as its target set. Given a number as an argument, the value of the function at this number is obtained as follows: Divide the argument by 20, multiply the result by 5 and add 1 at the end.
Does this function look familiar to you?

Example 6.3.3 (from geometry: central dilation in detailed notation)**.** In detailed notation, the map "central dilation at the origin $(0, 0)$ of the coordinate system with scale factor 2" is written as follows (if we call this map s):

$$s\colon \mathbb{R}^2 \to \mathbb{R}^2,$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

In this context, a point $(x, y) \in \mathbb{R}^2$ of the plane is usually written as a "column vector" $\begin{pmatrix} x \\ y \end{pmatrix}$.

Exercise E4 Describe the following maps (= functions) in a similar way to Example 6.3.3.

(a) Point reflection p at the origin.
(b) Translation v by 2 units to the right and by 3 units upwards.
(c) Reflection r at the x-axis.
(d) Reflection r' at the y-axis.
(e) Reflection r'' at the "angle bisector of the first quadrant" (= «erste Winkelhalbierende») (= the angle bisector of the two coordinate axes that passes through the point $(1, 1)$).
(f) Rotation d around the origin of the coordinate system by 90° in the mathematically positive direction of rotation.

Bonus: If you like, you can describe other geometric mappings in a similar way, for example:

- central dilation at $(2, 3)$ wit scale factor 7;
- point reflection at $(2, 3)$;
- reflection at a line that is parallel to the x-axis;
- Rotation by 60° around the origin or around some other point.

Test your solutions by plugging in some points!

Exercise E5 Someone tries to define three functions as follows.

$$f\colon \mathbb{R} \to \mathbb{R} \qquad g\colon \mathbb{R}_0^+ \to \mathbb{Z} \qquad h\colon \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \sqrt{x} \qquad y \mapsto \sqrt{y-1} \qquad z \mapsto \tfrac{1}{z+1}$$

Why are all these definitions wrong? (We say that these three functions are not "well defined".)
Change source and target sets in the definitions as little as possible so that the definitions become correct.

6.4 Graph of a function

6.4.1. A graph is commonly used to give an intuitive picture of a function.

This applies in particular to so called **real-valued functions of one real variable** or **real functions**, i. e. functions between subsets of $\mathbb{R}$. This is the type of function we will mostly consider in the near future.

Example 6.4.2. Every time temperature diagram in a weather forecast is a graph. It represents the function T that maps a time t to the temperature $T(t)$ at this time. Time t is usually plotted horizontally (as the t-axis), temperature vertically (as the T-axis).

✎Draw a time temperature diagram for some day, e. g. St. Gallen today, see Meteo-Schweiz; preverably a winter day with temperature around 0 °C. Mark point $(2, T(2))$ and a general point $(t, T(t))$ with coordinates marked on the axes.

Exercise E6

(a) Draw a coordinate system with a horizontal t-axis (label t instead of x) and a vertical s-axis (label s instead of y). At time $t = 0$ the "Trogenerbahn" departs from St. Gallen main station. Draw the distance traveled $s(t)$ as realistically as possible for each time t, up to the Spisertor stop.
Recommendation: Time t in minutes from -1 to 5, 1 cm corresponds to half a minute; distance s in m fom 0 to 1200, 1 cm corresponds to 200 m.

(b) Now consider the vertical axis as the v-axis and plot the velocity $v(t)$ at each time t (preferably in a new color).
Recommendation: Velocity v in $\frac{\text{km}}{\text{h}}$, 1 cm corresponds to 10 $\frac{\text{km}}{\text{h}}$.

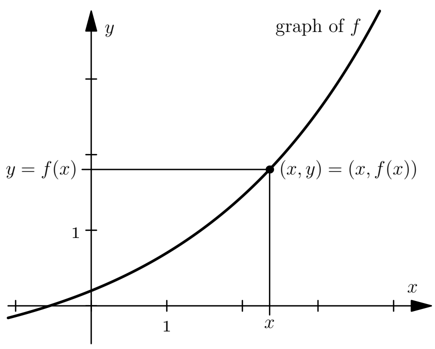
(c) What is the relation between the "velocity curve" drawn in part (b) and the "distance/position curve" drawn in part (a)?

6.4.3. In Exercise E6 you have drawn the graphs of the function $s = s(t)$ und $v = v(t)$.

Definition 6.4.4 Graph of a real function

The **graph of a function** $f\colon X \to Y$ between sets X and Y of real numbers is the subset of $\mathbb{R}^2$ defined by ✎

$$\begin{aligned}\text{graph}(f) &:= \left\{(x, f(x)) \in \mathbb{R}^2 \mid x \in X\right\}\\ &= \left\{(x, y) \in X \times \mathbb{R} \mid y = f(x)\right\} \subset \mathbb{R}^2\end{aligned}$$



Since $\mathbb{R}^2$ is the coordinate plane, the graph of f can be obtained as follows:

- For each $x \in X$, compute $f(x)$ and mark the point $(x, f(x))$.

The set of points in the plane obtained in this way is the graph of f.

The picture shows the graph of a function: If you imagine the point x on the x-axis as a variable point, the graph illustrates how the corresponding function value $f(x)$ changes.
This way of drawing the graph is also called a **plot** of the function.

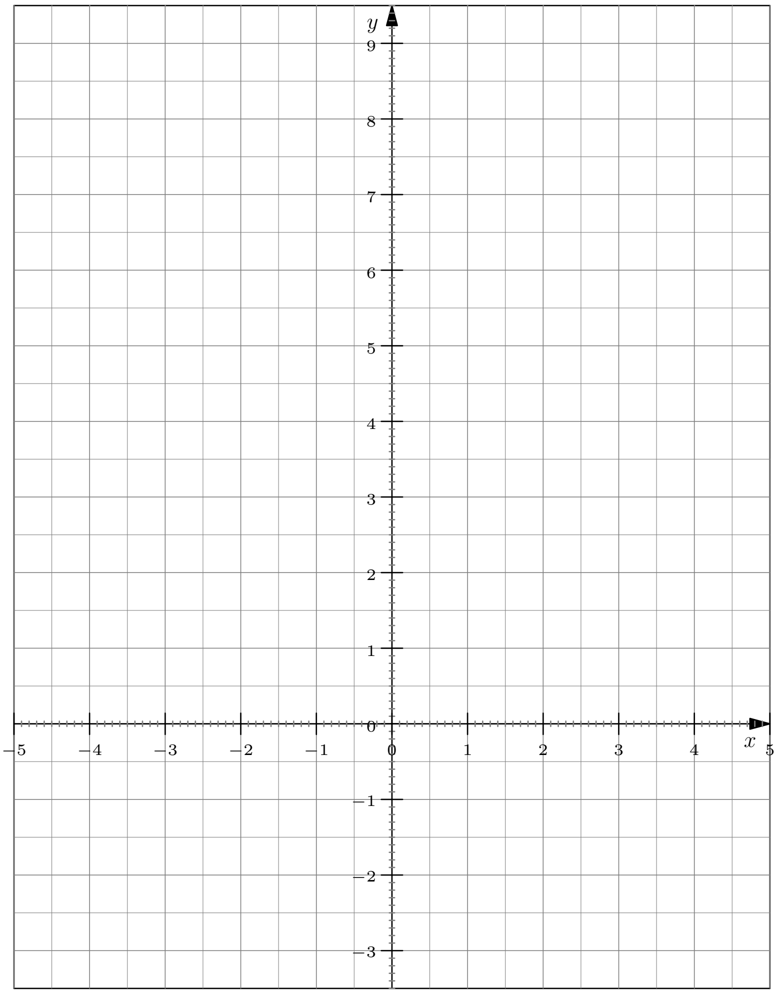
More generally, the graph of a function $f\colon X \to Y$ between arbitrary sets is the subset $\text{graph}(f) := \left\{(x, f(x)) \in X \times Y \mid x \in X\right\} = \left\{(x, y) \in X \times Y \mid y = f(x)\right\} \subset X \times Y$.

Exercise E7 In this exercise, you will learn how the graphs of (some of) the most important mathematical functions look like. You will encounter these function often in the future. When they appear, you should immediately have an idea of what their graphs look like.

(a) Draw the graphs of the following functions in the coordinate system.

(To get a good idea how the graphs look like, you should at least calculate the values of these functions at all integers $x \in \mathbb{Z}$ and for some values close to zero such as $\frac{1}{2} = 0.5$, $\frac{1}{4} = 0.25$, $\frac{1}{10} = 0.1$.)

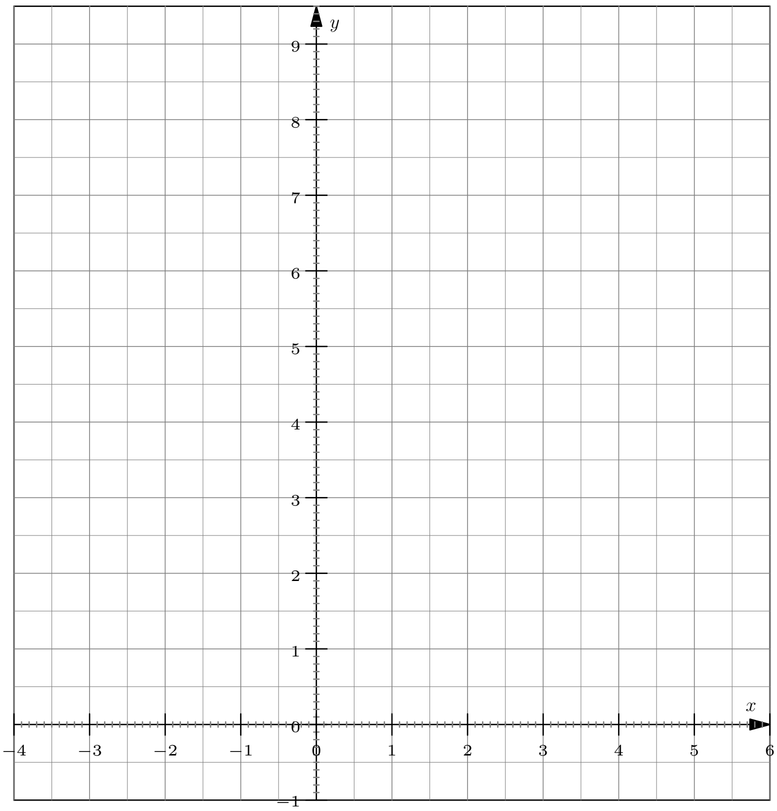
- $k(x) = x^3$
- $q(x) = x^2$ parabola
- $\ell(x) = x^1 = x$
- $c(x) = x^0 = 1$
- $h(x) = x^{-1} = \frac{1}{x}$ hyperbola



(b) Draw the graphs of the following functions in the coordinate system.

- $w(x) = \sqrt{x}$ square root (function)
- $b(x) = |x|$ absolute value (function)
- $z(x) = 2^x$ exponential function with base 2
- $d(x) = \left(\frac{1}{3}\right)^x$ exponential function with base $\frac{1}{3}$

Strictly speaking, you can only evaluate the last two functions for integers $x \in \mathbb{Z}$; trust your calculator!



Exercise E8 Consider the function $f(x) = x^2 - 2x - 8$.

(a) Does the point $(3, -4)$ belong to the graph of f?

(b) Is $(-3, 7)$ an element of the graph of f?

(c) For which $y \in \mathbb{R}$ does $(2, y) \in \operatorname{graph}(f)$ hold true?

(d) Where do the graph of f and the y-axis intersect?

The solution to each of the following two questions is an equation. It is not necessary to solve this equation, but if you want to: All solutions are integers in the interval $[-10, 10]$.

(e) Which equation should be solved to find an x with $(x, 16) \in \operatorname{graph}(f)$?

(f) Which equation would you solve to find the points of intersection of the x-axis and the graph of f?

(g) Try to find out what should be added to the following memo box 6.4.5.

Important 6.4.5

Let f be a function.

A point $P = (x_P, y_P)$ belongs to (= lies on) the graph of f $\iff$ ✎ $f(x_P) = y_P$.

The graph of f intersects the x-axis at x_0 $\iff$ ✎ $f(x_0) = 0$.

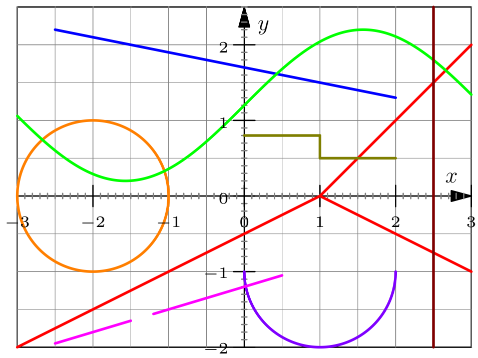
To find the intersection of the graph of f and the x-axis, ✎ solve the equation $f(x) = 0$.

The graph of f intersects the y-axis at ✎ f(0).

Exercise E9

Decide for each color (orange, green, red, blue, purple, magenta, olive, brown):
Is the set of all points of this color the graph of a function or not.

- If it is not the graph of a function, why not?
- If it is the graph of a function: Write down its domain $\mathbb{D}$ (as a subset of the interval $[-3, 3]$, because more cannot be determined from the picture).



Definition 6.4.6 zero of a function (Nullstelle)

If f is a real-valued function, any element x of the domain of f satisfying

$$f(x) = 0 \qquad \text{(i. e. } f \text{ **vanishes at** } x)$$

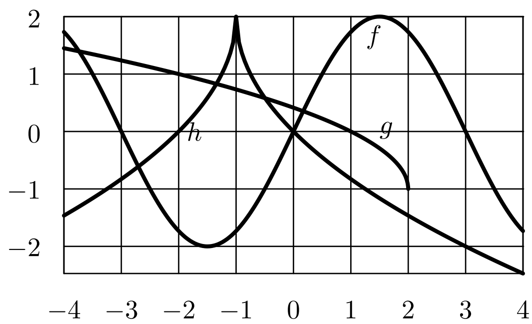
is called a **zero of** f (Nullstelle).

This terminology comes from the fact that f has the value zero at such an element.
Sometimes, the old-fashioned term “root of a function” is used instead of “zero of a function”.

Exercise E10

(a) From their graphs, read of the function values of f, g and h at all integers (ganze Zahlen) between -4 and 4. Remark: Expressions for the functions are not required here.

(b) For each of the functions f, g and h find all zeros (that are visible in the picture).



Definition 6.4.7 maximal set of definition of a function

If a function f is given in short form as

$$f(x) = (\text{defining expression}) \qquad \text{e.g. } f(x) = \sqrt{3-x}$$

its **(maximal) set of definition** or **(maximal) domain** $\mathbb{D}$ is the largest subset of $\mathbb{R}$ for which the defining expression can be evaluated, i. e.

$$\mathbb{D} = \{x \in \mathbb{R} \mid f(x) \text{ can be calculated/is defined}\}$$

Important 6.4.8

If a function is given in short form, the author hopes that the reader is intelligent enough to find out by himself what the domain (= maximal domain) of the function is.

Example 6.4.9. For $f(x) = \sqrt{3-x}$, find the maximal set of definition $\mathbb{D}$. ✎

For the expression $\sqrt{3-x}$ to be defined, we must have

$$3 - x \geq 0 \iff 3 \geq x \iff x \leq 3$$
$$\implies \mathbb{D} = (-\infty, 3]$$

Example 6.4.10. For $f(x) = \frac{4x-3}{2x+3}$, find the maximal set of definition $\mathbb{D}$. ✎

The "bad case" is that the denominator is zero:

$$2x + 3 = 0 \iff 2x = -3 \iff x = -\frac{3}{2}$$
$$\implies \mathbb{D} = \mathbb{R} \setminus \left\{-\tfrac{3}{2}\right\}$$

Example 6.4.11. For $f(x) = \sqrt{5-x} + \frac{1}{x-3} + \frac{1}{\sqrt{x}}$, find the maximal set of definition $\mathbb{D}$. ✎

We need to have

$$5 - x \geq 0 \quad \text{and} \quad x - 3 \neq 0 \quad \text{and} \quad \sqrt{x} \neq 0 \quad \text{and} \quad x \geq 0$$
$$\iff 5 \geq x \quad \text{and} \quad x \neq 3 \quad \text{and} \quad x > 0$$

$$\implies \mathbb{D} = (-\infty, 5] \cap \left(\mathbb{R} \setminus \{3\}\right) \cap \mathbb{R}^+ = (0, 5] \setminus \{3\}$$

Exercise E11 Find the maximal domain $\mathbb{D}$ of the following functions given in short form.
Hints: Binomial formulas; Pascal's triangle; a product is zero if and only if at least one factor is zero.

a) $f(x) = \frac{1}{x}$
b) $f(x) = \sqrt{x}$
c) $f(x) = \frac{1}{x-2}$
d) $f(x) = \sqrt{-x}$
e) $f(x) = \frac{1}{x+5}$
f) $f(x) = \sqrt{x+5}$

g) $f(x) = \frac{1}{2x-5}$

h) $f(x) = \sqrt{x^2+1}$

i) $f(x) = \frac{1}{(x-2)(x+2)}$

j) $f(x) = \sqrt{-x+5}$

k) $f(x) = \frac{1}{x^2-6x+9}$

l) $f(x) = \sqrt{16-x^2}$

m) $f(x) = \frac{1}{x^4+4x^3+6x^2+4x+1}$

n) $f(x) = \sqrt{x^2-16}$

o) $f(x) = \sqrt{2+x} + \sqrt{2-x}$

p) $f(x) = \frac{1}{\sqrt{2+x}} + \frac{1}{\sqrt{2-x}}$

Definition 6.4.12 range (= image set) of a function

If $f\colon X \to Y$ is a function, the set

$$\text{range}(f) = f(X) := \{f(x) \mid x \in X\}$$ eventuell diese Notation auch einführen

is called the **range** or **image (set)** of the function f (Bild oder Bildmenge).
The range of f consists of all elements y of the target set Y that are "hit by the function": there is an element $x \in X$ such that $f(x) = y$.

Exercise E12 For each of the functions given below:

- Find and write down the maximal set of definition $\mathbb{D}$ of the function.
- Draw a (small) coordinate system and use a color to highlight the set of definition $\mathbb{D}$ on the horizontal axis. The length of the horizontal axis should be about 6 cm.
- From the graph, read of the image set of the function and write it down in mathematical notation (no exact justification required).
  Note: The image set of a function $g\colon \mathbb{D} \to Y$ is denoted by $g(\mathbb{D})$, i. e. "function(domain)".

a) $e(u) = -u$

b) $f(v) = \frac{1}{2}v - 1$

c) $g(w) = \sqrt{9-w^2}$

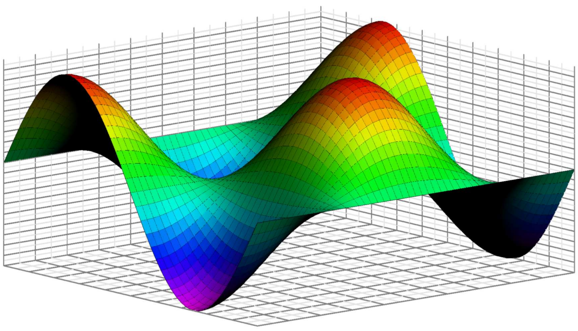
d) $h(y) = \sqrt{-y}$

e) $i(x) = -|x-1| + 1$

6.4.13. Graphs of functions can also be drawn using GeoGebra or other Computer Algebra Systems (CAS). Online, this can be done using Geogebra Calculator or Desmos Grafik-Rechner, for example.

Initially, you should use these tools only for checking your solutions, but not so solve them, as such tools will not be allowed in the exam.

6.4.14. As mentioned, we will mainly consider graphs of functions between subsets of $\mathbb{R}$. However, it should be mentioned that many other functions can also be represented by graphs, for example functions from subsets of the coordinate plane $\mathbb{R}^2$ to the real line $\mathbb{R}$.



The picture shows the graph of the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}$$
$$(x,y) \mapsto f(x,y) = \cos(2\pi x)\cdot\sin(2\pi y)$$

(The colors indicate the height (= z-coordinate) of the corresponding point and are only used to make the graph look nicer.)

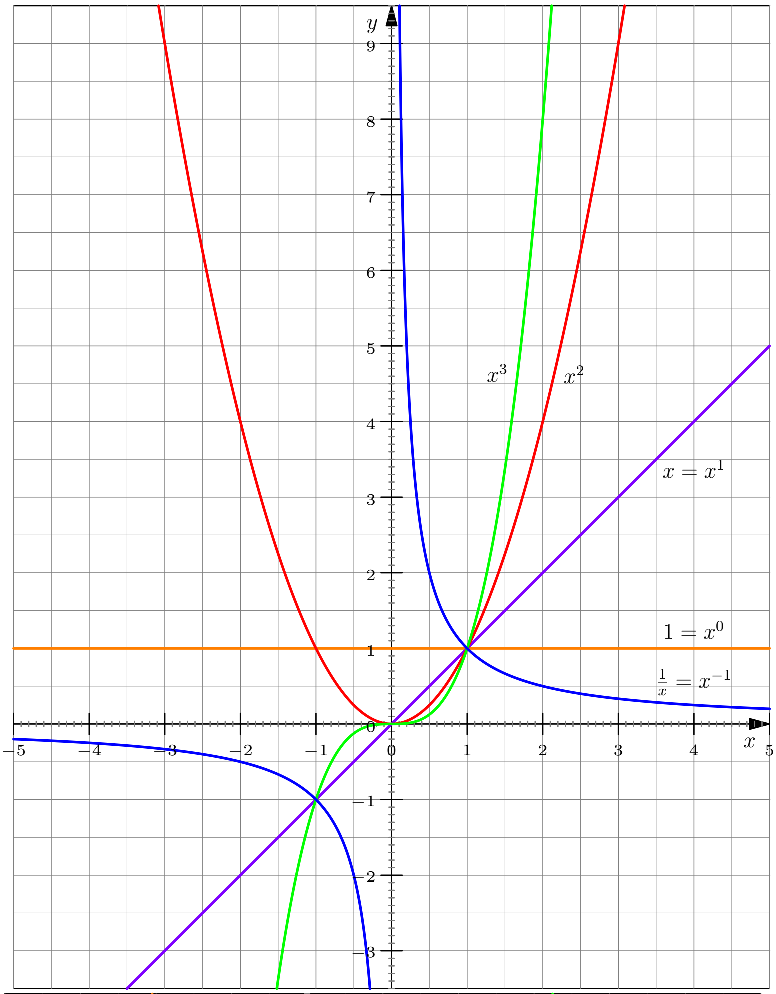
Source: Asymptote, Gallery, 3D Graphs, https://asymptote.sourceforge.io/, slightly modified.

Another example is the height function h that maps each point (x,y) of Switzerland to its height $h(x,y)$. The graph of h is a "raised relief map" of Switzerland (cf. Wikipedia: Raised-relief map).

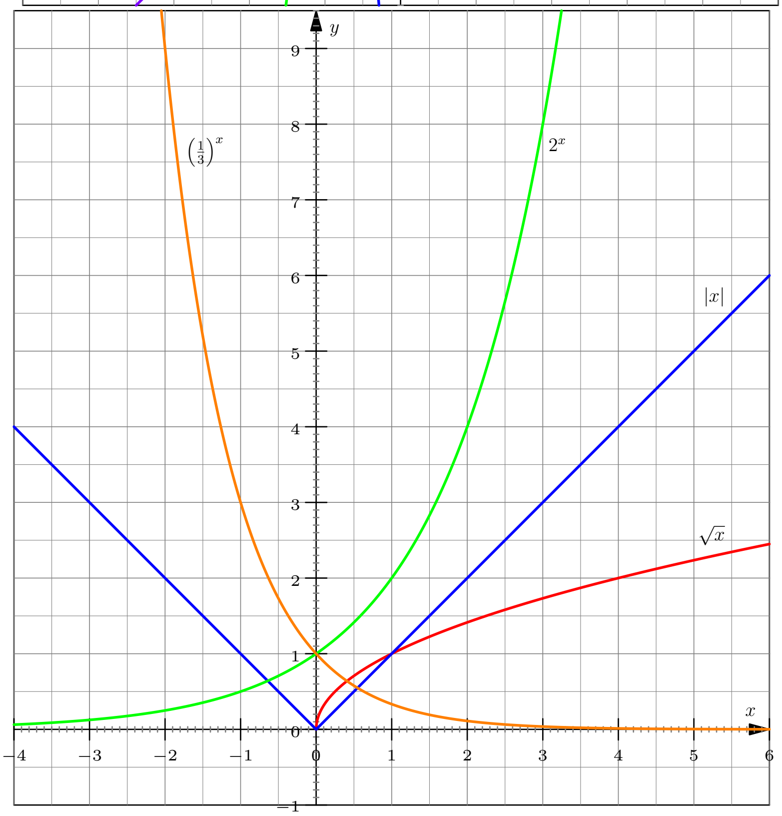
We could also consider the temperature function T that maps each point (x,y) of a country to its temperature $T(x,y)$. The corresponding graph is often displayed in color in weather forecasts; different colors represent different temperatures. Many similar graphs can be found on Meteo Schweiz.

Important 6.4.15 Graphs of important functions – remember how they look like! (Solution of E7

- $x^{-1} = \frac{1}{x}$ hyperbola
- $x^0 = 1$ constant function
- $x^1 = x$ identity function: the value of the function is identical to the argument: x is mapped to x.
- x^2 parabola; note that it is smooth and has a horizontal tangent at $x = 0$
- x^3 it is also smooth and has a horizontal tangent at $x = 0$



- $|x|$ absolute value function, note the kink at $x = 0$
- $\sqrt{x}$ square root function
- 2^x exponential growth with growth factor 2: The value of the function is multiplied by 2 whenever the argument increases by 1: $2^{x+1} = 2^x \cdot 2^1 = 2 \cdot 2^x$.
- $\left(\frac{1}{3}\right)^x$ exponential growth with growth factor $\frac{1}{3}$: The value of the function is multiplied by $\frac{1}{3}$ whenever the argument increases by 1: $\left(\frac{1}{3}\right)^{x+1} = \left(\frac{1}{3}\right)^x \cdot \left(\frac{1}{3}\right)^1 = \left(\frac{1}{3}\right) \cdot \left(\frac{1}{3}\right)^x$.



6.5 Lines and linear functions

6.5.1. By a "line" we usually mean a "straight line" in the following.

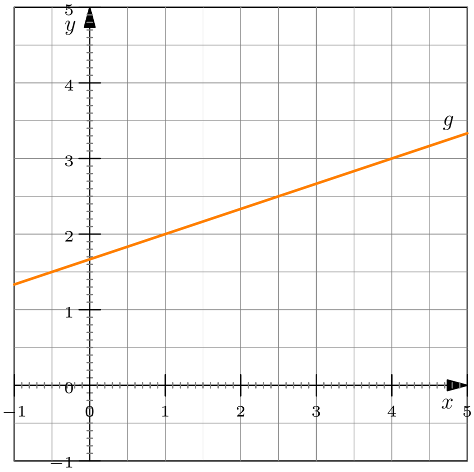
Definition 6.5.2 slope, y-intercept

The **slope** m of a line g in the coordinate plane that is not parallel to the y-axis is calculated as follows:

- Choose two distinct points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ on the line g.
- Then the slope is defined as follows:

slope $= m =$ ✎ $\dfrac{\Delta y}{\Delta x}$

$$= \frac{y_B - y_A}{x_B - x_A} = \frac{3-2}{4-1} = \frac{1}{3}$$
$$= \frac{y_A - y_B}{x_A - x_B} = \frac{2-3}{1-4} = \frac{-1}{-3} = \frac{1}{3}$$



Here, Δx stands for "difference of x-values", Δy stands for "difference of y-values". The greek letter Delta Δ corresponds to our D and stands for *difference*.
Any right-angled triangle with legs Δx and Δy as in the picture is called a **slope triangle**. Add Δx and Δy as arrows, decomposing the arrow $\overrightarrow{AB}$!

The y**-intercept** q of a line g is the y-coordinate of the point where g intersects the y-axis.
(Steigung, Steigungsdreieck, y-Achsenabschnitt)

Note: The slope m does *not* depend on the choice of the two points A and B because all slope triangles are similar to one another.
Note: The slope can be positive, negative or zero.

- If the line goes up, then its slope is positive. (The steeper the line, the bigger the slope.)
- If the line goes down, its slope is negative.
- If the line runs horizontally (= parallel to the x-axis), its slope is zero.

Important 6.5.3

Consider a point moving on a line. Then the slope m tells you how much the y-coordinate increases if its x-coordinate increases by 1.
To see this, just consider any slope triangle with horizontal leg $\Delta x = 1$, then $m = \frac{\Delta y}{\Delta x} = \frac{\Delta y}{1} = \Delta y$.

Exercise E13 In one coordinate system, draw 11 lines through the origin with slopes $\pm\frac{1}{4}$, $\pm\frac{1}{2}$, ±1, ±2, ±4 und 0. Label each line with its slope.

Definition 6.5.4 slope angle

The **slope angle** (or **angle of inclination**) of a line in a coordinate plane is the angle between the x-axis and the line (in the mathematically positive direction of rotation). It is usually an element of the interval $(-90°, 90°]$.
(Steigungswinkel)

Exercise E14 For each of the following angles, determine **precisely** (using the Pythagorean theorem) the slope of the line with the given slope angle.
(Do not read off the values of Δx and Δy from a slope triangle, as this is generally imprecise.)

a) 0° b) 30° c) 45° d) 60° e) 90°

Example 6.5.5. The slope (or gradient) of a road is usually expressed as a percentage. Examples:

- The number 20% on a road sign means that the road has an average slope of $20\% = 20 \cdot \frac{1}{100} = \frac{20}{100} = 0.2$. For every 1 km of horizontal travel, you move 0.2 km=200 m upward; the slope angle is then approximately $12.6°$.
- A (hypothetical) road with a slope $100\% = 100\frac{1}{100} = 1$ has a slope angle of $45°$.
- A (hypothetical) road with a slope $200\% = 200\frac{1}{100} = 2$ has a slope angle of approximately $70.5°$.

Definition 6.5.6 linear function

A function $f\colon \mathbb{R} \to \mathbb{R}$ is called **linear** if it can be written as

$$f(x) = mx + q$$

where m and q are suitable real numbers.

At university level, such functions are called affine-linear.

Example 6.5.7. The standard grading function $n(p) = \frac{5}{a}p + 1$ is linear in the variable p, where a is the number of points required for grade 6.

Example 6.5.8. A cyclist is riding at constant speed v. Find the function that describes the distance $s(t)$ traveled at time t. ✎$s(t) = vt$, linear function in t

Example 6.5.9. A steady flow of 200 L water per minute enters a swimming pool. Find the function $V(t)$ that describes the volume of water (in liters) at the time t (in minutes), given that

- the pool is empty at time $t = 0$. ✎$V(t) = 200t$, linear function in t
- the pool already contais 2500 L of water at time $t = 0$. ✎$V(t) = 200t + 2500$, linear function in t

Exercise E15 Plot the graphs of the following linear functions on the same coordinate system. Is the graph a straight line? If yes, determine slope and y-intercept.

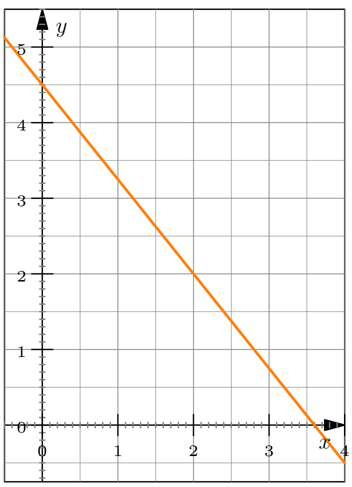
a) $a(x) = 3x$
b) $b(x) = \frac{1}{3}x$
c) $c(x) = -x$
d) $d(x) = -\frac{1}{2}x$
e) $e(x) = x - 1$
f) $f(x) = -x + 1$
g) $g(x) = 2x - 2$
h) $h(x) = -\frac{1}{3}x + \frac{2}{3}$
i) $i(x) = -3$
j) $j(x) = 0$

Theorem 6.5.10 non-vertical lines = graphs of linear functions

Non-vertical lines are graphs of linear functions.
More precisely: Let m and q be arbitrary real numbers. Then the following two subsets of the coordinate plane coincide:

- the line g with slope m and y-intercept q;
- the graph $\operatorname{graph}(\ell)$ of the linear function $\ell(x) = mx + q$.

Proof. Certainly, $A = (x_A, y_A) = (0, \ell(0)) = (0, q)$ is the only point on the y-axis that lies on g and on $\operatorname{graph}(\ell)$. Let $B = (x_B, y_B)$ be an arbitrary point that is not on the y-axis, i. e. $x_B \neq 0$. Then ✎



$$\begin{aligned}
B \in g \quad &\iff \quad m = \frac{y_B - y_A}{x_B - x_A} = \frac{y_B - q}{x_B - 0} = \frac{y_B - q}{x_B} \\
&\iff \quad mx_B = y_B - q \\
&\iff \quad mx_B + q = y_B \\
&\iff \quad \ell(x_B) = y_B \\
&\iff \quad B \in \operatorname{graph}(\ell)
\end{aligned}$$

This shows the claimed equality

$$g = \operatorname{graph}(\ell)$$ □

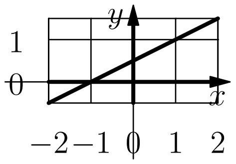
Ivo Blöchliger, Olaf Schnürer

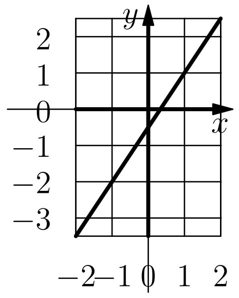
Corollary 6.5.11 Plotting graphs of linear functions

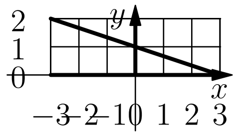
In order to plot the graph of a linear function $f(x) = mx + q$, you may either

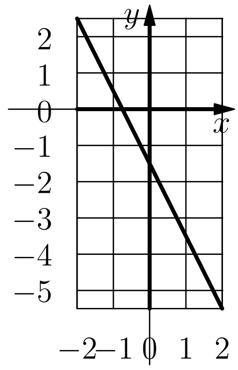
- calculate two distinct points on the graph and connect them by a straight line (one obvious point is $(0, q)$), or
- draw the line through the point $(0, q)$ (or any other point on the graph) with slope m.

Exercise E16 For each of the following lines, find the linear function that has this line as its graph.

a)


b)


c)


d)


Definition 6.5.12 linear relationship, proportional

Two variables x and y are said to be **in a linear relationship** if y is a linear function of x, i. e.

$$y = mx + q$$

The set of all points (x, y) satisfying this equation is then the line with slope m and y-intercept q.
If, moreover, the y-intercept is zero, i. e. $q = 0$, we say that x and y are **proportional**, i. e.

$$y = mx$$

The set of all points (x, y) satisfying this equation is then the line *through the origin* with slope m (Ursprungsgerade). The slope $m = \frac{y}{x}$ is then also called the **factor/constant of proportionality** (Proportionalitätsfaktor).

Examples 6.5.13.

- The circumference $U = U(r)$ of a circle is proportional to its radius r,

$$U = U(r) = 2\pi r \qquad \text{i. e. the factor of proportionality is } \frac{U(r)}{r} = 2\pi$$

- The mass m of a substance is proportional to its volume V, because a doubling (tripling, ...) of the volume leads to a doubling (tripling, ...) of the mass.
  The corresponding constant of proportionality is the (mass) density of the substance and is usually denoted with the lowercase Greek letter ρ ("rho"), i. e.

$$\rho = \frac{m}{V}$$

  For example, iron (Latin *ferrum*, element symbol Fe) has density

$$\rho_{\text{Fe}} = 7874 \frac{\text{kg}}{\text{m}^3}$$

  Specifically, one liter (= cubic decimeter = $(10\,\text{cm})^3 = 1000\,\text{cm}^3$) iron weighs

$$m = \rho_{\text{Fe}} \cdot V = 7874 \frac{\text{kg}}{\text{m}^3} \cdot 1\,\text{dm}^3 = 7874 \frac{\text{kg}}{\text{m}^3} \cdot 10^{-3}\text{m}^3 = 7.874\,\text{kg}$$

- The price of a product is often proportional to the quantity (unless there are discounts, etc.). Examples: gold price, wheat price, milk price, etc.
- If a cyclist rides at constant speed v, the distance traveled $s = s(t) = vt$ is proportional to time t. The factor of proportionality is the constant velocity v.

$$v = \frac{s}{t} \qquad \text{or} \qquad s = s(t) = vt$$

6.5.14. We mention the following terminology for completeness: We say that two variables x and y are **inversely proportional** (antiproportional) if $\frac{1}{x}$ and y are proportional, i. e. $y = c\frac{1}{x}$ or equivalently $xy = c$ for some fixed real number c. For example, the number of workers and the time to finish a certain job may be inversely proportional: If 1 worker needs 30 days to finish the job, 2 workers need 15 days, 30 workers need 1 day. Of course, this example is only somewhat realistic.

Standard problems for lines

Example calculations? Line through two points: Conversion from Fahrenheit to Celsius. Picture of a thermometer with two scales on Wikipedia.

6.5.15. A line is uniquely determined in the following two ways:

(1) by two distinct points on the line;
(2) by a point on the line and its slope.

In the following two exercises, you will learn how to write down the corresponding equations.

Exercise E17 **Equation of a line through a given point with a given slope**

(a) Using specific numbers: Determine the equation $y = \ell(x) = mx + q$ of the line with slope $m = -\frac{2}{3}$ that passes through the point $A = (-1, -3)$.
Once you have determined the equation:
- Check whether the point A lies on the graph of $\ell = \ell(x)$.
- Does the point $(4, 4)$ lie on the line?
- For what value of y does the point $(3, y)$ lie on the line?

(b) Abstractly with parameters: What should be added in memo box 6.5.16?
Check your guess:
- In an example: If you substitute for $A = (x_A, y_A)$ and m the values from part (a), do you recover the equation you found there?
- Abstractly: Is your function linear? Is m the coefficient at x? Does A lie on its graph?

Important 6.5.16 Equation of the line given by a point and a slope

The line through the point $A = (x_A, y_A)$ with slope m has the equation (resp. is the graph of the following function)

$$y = \ell(x) = ✎ m(x - x_A) + y_A \text{ maybe don't write down at all or maybe during the proof: } = mx \underbrace{-mx_A + y_A}_{y\text{-intercept } q\text{, real number}}$$

Proof. The given equation is linear and therefore describes a line. Its slope is the coefficient at x, which is m (as it should be). It remains to check that A lies on the graph of ℓ. But this is the case since ✎

$$\ell(x_A) = m(x_A - x_A) + y_A = 0 + y_A = y_A$$ □

Exercise E18 **Equation of a line through two points**

(a) Using specific numbers: Find the equation $y = \ell(x) = mx + q$ of the line that passes through the two points $A = (-1, 2)$ and $B = (5, -1)$.
Hint: The slope of this line is easy to find.
Check your equation: Is your function linear? Do the two points A and B lie on the graph of your function $\ell = \ell(x)$?

(b) Abstractly with parameters: What should be added in memo box 6.5.17?
Check your guess:
- In an example: What do you get by plugging in the values from part (a) for A and B?
- Abstractly: Is your function linear? Do A and B lie on its graph?

Important 6.5.17 Equation of the line through two points

The line through the two points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ with $x_A \neq x_B$ has the equation

$$\ell(x) = ✎ \underbrace{\frac{y_B - y_A}{x_B - x_A}}_{\text{slope}} \cdot (x - x_A) + y_A$$

If you swap the roles of A and B, you get the equation $\ell(x) = \frac{y_A - y_B}{x_A - x_B} \cdot (x - x_B) + y_B$. Its right-hand side coincides with the right-hand side given above. Why?

Proof. With a little experience, you can easily figure this out in your head: The equation is linear in x, has the required slope and passes through A. Here is the detailed argument: ✎

$$\ell(x) = \frac{y_B - y_A}{x_B - x_A} \cdot (x - x_A) + y_A = \underbrace{\frac{y_B - y_A}{x_B - x_A}}_{\text{real number, slope } m} \cdot x + \underbrace{\left(-\frac{y_B - y_A}{x_B - x_A} \cdot x_A + y_A\right)}_{\text{real number, } y\text{-intercept } q}$$

So ℓ is a linear function; therefore, its graph is a line.

$$\ell(x_A) = \frac{y_B - y_A}{x_B - x_A} \cdot (x_A - x_A) + y_A = 0 + y_A = y_A \implies A \text{ on graph}(\ell)$$

$$\ell(x_B) = \frac{y_B - y_A}{x_B - x_A} \cdot (x_B - x_A) + y_A = y_B - y_A + y_A = y_B \implies B \text{ on graph}(\ell)$$

If it is clear to you that the slope is correct, it is sufficient to check that one of the two points A, B belongs to the graph. □

Exercise E19 Consider two linear functions $f(x) = -\frac{1}{3}x - 1$ and $g(x) = \frac{3}{2}x + 1$.

a) Plot the graphs of f and g on a coordinate system with a scale of 1 unit per 6 squares (= Kästchen, Häuschen).

b) Read off the approximate coordinates of the point of intersection of the two lines.

c) Calculate the exact point of intersection (i. e. find both coordinates).
Hint: The point of intersection has both the form $(x_0, f(x_0))$ and the form $(x_0, g(x_0))$, for a suitable real number x_0. Why is this so, and why is this helpful?

Important 6.5.18 Calculation of the points of intersection of two graphs

The x-coordinates of the intersection points of the graphs of two functions f and g are obtained by solving the equation

$$f(x) = g(x)$$

for x.
If f and g are linear functions (with lines as graphs), then this equation is linear and can easily be solved.

Exercise E20 **Slopes of perpendicular lines**

(a) Example with numbers: Consider the line g through the origin with slope $m_g = \frac{3}{5}$. Let h be the line through the origin that is perpendicular to g. What is the slope m_h of this line h?
Hint: Draw a sketch with a slope triangle.

(b) If an arbitrary line has slope $\frac{3}{5}$, what is the slope of a line perpendicular to it?

(c) Abstractly, with parameters: What should be added in memo box 6.5.19?

Important 6.5.19 perpendicularity (= orthogonality, right angularity) and slopes of lines

Two lines (neither of which vertical) are perpendicular (= orthogonal) to each other if and only if ✎the product of their slopes is -1.

In formulae: For two non-vertical lines g and h with slopes m_g and m_h, the following holds. ✎

$$g \perp h \qquad \Longleftrightarrow \qquad m_g \cdot m_h = -1 \qquad \Longleftrightarrow \qquad m_g = -\frac{1}{m_h}$$

Alternatively: If a line has slope m, then every line perpendicular to it has slope ✎

$$m_\perp = -\frac{1}{m}$$

Applications

Exercise E21 The distance (by rail) from Wattwil to Kaltbrunn is 8 km.
The express train from Wattwil to Kaltbrunn travels at a speed of 90 km/h. It departs from Wattwil at 9:02.
In the opposite direction, the regional train from Kaltbrunn to Wattwil departs at 9:00 and travels at 60 km/h.
We neglect the trains' acceleration and deceleration phases and assume that each train always travels at the same speed.

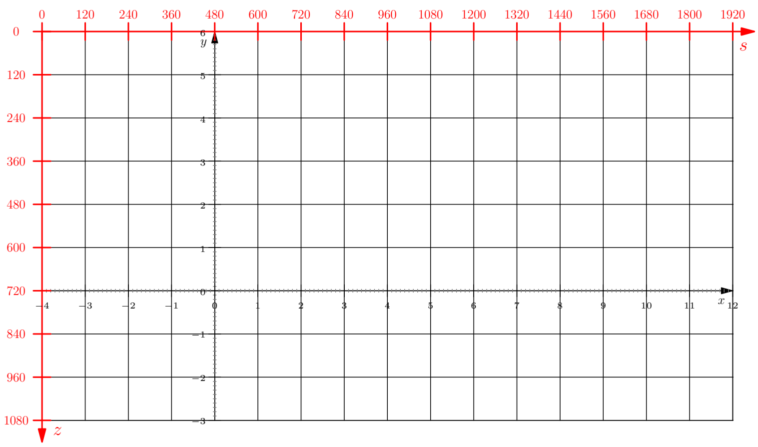
(a) Draw a distance-time diagram (horizontal axis: t = time in minutes after 9:00; vertical axis: s = distance in kilometers where Wattwil is at $s = 0$ and Kaltbrunn is at $s = 8$).
Determine graphically when the two trains meet.

(b) Determine by calculation when this occurs.
Hint: Find linear functions $s = s(t)$ (express train = Schnellzug) and $r = r(t)$ (regional train) such that
- $s(t)$ = position of the express train at time t and
- $r(t)$ = position of the regional train at time t.

Exercise E22 A laptop screen is 1920 pixels wide and 1080 pixels tall (with quadratic pixels; so the width-to-height-ratio is $\frac{1920}{1080} = \frac{16}{9} = 16 : 9$). The standard s-z coordinate system on such a screen looks as follows (shown in red in the picture):

- horizontal coordinate s (Spalte), ranging from 0 to 1920; the s-axis points to the right;
- vertical coordinate z (Zeile), ranging from 0 to 1080; **however, the z-axis points downward!** (row 0 is at the very top, row 1080 at the very bottom).

Someone now wants to use the "standard x-y coordinate system" on such a screen, with x-coordinates from -4 to 12 and y-coordinates from -3 to 6, as indicated in the picture.



(a) Consider the points $U = (0, 0)$, $A = (12, 6)$ and $B = (-4, -3)$ in the x-y coordinate system. What are their coordinates in the s-z coordinate system?

(b) Let $s(x)$ be the linear transformation function that calculates the s-coordinate of a point from its x-coordinate. For example, $s(12) = 1920$.
- What is $s(0)$? What is $s(-4)$?
- Plot the graph of $s = s(x)$ in an x-s diagram (horizontal x-axis, vertical s-axis).
- Determine the function $s(x)$.
- Test your function! Is it linear? Do $s(12)$ and $s(-4)$ yield the correct values?

(c) Find the linear function $z(y)$ that calculates the z-coordinate of a point from its y-coordinate!
Test your result by calculating $z(-3)$, $z(0)$ and $z(6)$.

Exercise E23 Consider the linear functions $f(x) = 2x - 1$ and $g(x) = 2x + 1$.

(a) Plot the graphs of these two functions.
(b) Measure the distance between the two lines (in units, not in cm).
(c) Calculate the exact distance betweeen these two lines.
Hint: Use an arbitrary line that is perpendicular to the two given lines.
(d) Calculate the distance between two parallel lines with the same slope m and difference Δq of y-intercepts.

Exercise E24 Three points $A = (-2, 1)$, $B = (1, -2)$ and $C = (3, 1)$ are given. They form a triangle ABC.

(a) Find the equations of the three lines $a = BC$, $b = AC$ and $c = AB$.
(b) Is $a \perp c$ (perpendicular)?
(c) Determine the equation of the height (= altitude) h_a (this is the line through A that is perpendicular to a).
(d) Determine the foot (Höhenfusspunkt) $H_a = a \cap h_a$ of the height h_a.

Exercise E25 Consider the two villages $A = (1, 2)$ and $B = (5, 3)$. Assume that the x-axis is a river.

(a) What is the length of the shortest path from village A to village B that includes a stop at the river?
Hint: Reflect one of the villages across the river. The problem can then be solved directly using the Pythagorean theorem (without equations of lines).
(b) Determine the point S on the river where the shortest path from above meets the river.

Note: The problem becomes somewhat more difficult if the river is given by any other straight line (such that the two villages lie on the same side of the river).

Exercise E26 Two points $A = (-2, -1)$ and $B = (3, 1)$ are given, as well as the slope $m = \frac{3}{2}$ of a line b through point A. We want to find the point C on b such that the triangle $\triangle ABC$ is perpendicular with right angle at C.

(a) Construct this triangle and read off the approximate coordinates of C.
(b) Calculate the exact coordinates of C.

Exercise E27 Consider the triangle with vertices $A = (2, 1)$, $B = (17, 4)$, $C = (8, 13)$.

(a) Find the equation of the line c through A and B. $c(x) = \frac{1}{5}x + \frac{3}{5}$
(b) Find the equation of the height (= altitude) h_c. $h_c(x) = -5x + 53$
(c) Find M_{AB} (= the midpoint of A and B), i, e. determine its coordinates. You will need this point in the following two subproblems.
Hint: How do you obtain the x-coordinate of M_{AB} from the x-coordinates of A and B?
$M_{AB} = \left(\frac{19}{2}, \frac{5}{2}\right) = (9.5, 2.5)$
(d) Find the equation of the perpendicular bisector m_c (= Mittelsenkrechte). $m_c(x) = -5x + 50$
(e) Find the equation of the median s_c (= Seitenhalbierende). $s_c(x) = -7x + 69$
(f) Find the equation of the angle bisector w_γ.
Hint: Angle bisector theorem (= Winkelhalbierendensatz). $w_\gamma(x) = -(3 + \sqrt{10})x + 37 + 8\sqrt{10}$

Exercise E28 By first considering what the graph of the function might look like, determine all linear functions that

(a) map the number 1 to 3 and the number 4 to 2.
(b) map the interval $[-1, 1]$ onto the interval $[0, 4]$.
(c) the interval $[0, 1]$ onto the interval $[1, 6]$.
(d) the interval $[1, 6]$ onto the interval $[0, 1]$.
(e) the interval $[0, 1]$ onto the interval $[a, b]$ (where $a < b$ are arbitrary real numbers).
(f) the interval $[a, b]$ onto the interval $[0, 1]$ (where $a < b$ are arbitrary real numbers).
(g) the interval $[a, b]$ onto the interval $[c, d]$ (where $a < b$ and $c < d$ are arbitrary real numbers).

Comprehension questions

Exercise E29 Find examples of linear functions in everyday life.

Exercise E30 Consider the function/line $\ell(x) = mx + q$.
(If you like, you can start by working with a specific line such as $\ell(x) = \frac{1}{2}x + 1$.)

(a) If you reflect this line across the x-axis, which linear function describes the reflected line?
(b) If you reflect this line across the y-axis, which linear function describes the reflected line?
(c) If you reflect this line across the first angle bisector (= erste Winkelhalbierende), which linear function describes the reflected line?
(d) If you rotate this line by 90° in the positive mathematical direction, which linear function describes the rotated line?

Exercise E31 True or false? Justify your answers and, if possible, correct false statements.

(a) For any two points in the x-y-plane, there is always a linear function whose graph passes through these two points.
(b) If two lines have the same slope, they are parallel.
(c) If two lines share the same y-intercept, they intersect on the x-axis.
(d) Given is a linear function $f(x) = mx + q$. If both m and q are replaced by their negatives, the graph is reflected across the y-axis.
(e) The slope of a horizontal line is not defined.
(f) The slope of the first angle bisector (= erste Winkelhalbierende) is 1.
(g) The slope of a vertical line is 2.
(h) The product of the slopes of two perpendicular lines is -1.
(i) If the slope of a line is increased by 2, the line shifts by 2 units in the y-direction.
(j) If f is a linear function, then $h(x) = 2 \cdot f(x)$ is also a linear function.
(k) Let f and g be linear functions. Then the function $h(x) = f(x) + g(x)$ is also a linear function.
(l) If f and g are linear functions, then $h(x) = f(g(x))$ is a linear function as well.
(m) If f is a linear function, then $g(x) = f(x) \cdot f(x)$ is a linear function as well.
(n) The graphs of the two functions $f(x) = x^2$ and $g(x) = x + 2$ intersect at the points $(-1, 1)$ and $(2, 4)$.
(o) The graphs of the functions $f(x) = (x + 1)^2 - 1$ and $g(x) = (x - 1)^2 + 1$ intersect at exactly one point.
(p) The graphs of the functions $f(x) = (x - 1)^2 + 1$ and $g(x) = 1 - 2x$ intersect at exactly two points.
(q) The sum $h(x) = f(x) + g(x)$ of two non-linear functions f and g is never linear.
(r) The product $h(x) = f(x) \cdot g(x)$ of two linear functions f and g is never linear.
(s) The distance between two lines is equal to the difference of their y-intercepts.
(t) For any real number x, the point $(f(x), x)$ lies on the graph of f.
(u) The standard grading function is a linear function of the score.

6.6 Lösungen

Hinweise zu den Symbolen:

Diese Aufgaben könnten (mit kleinen Anpassungen) an einer Prüfung vorkommen. Für die Prüfungsvorbereitung gilt: "If you want to nail it, you'll need it".

Diese Aufgaben sind wichtig, um das Verständnis des Prüfungsstoffs zu vertiefen. Die Aufgaben sind in der Form aber eher nicht geeignet für eine Prüfung (zu grosser Umfang, nötige «Tricks», zu offene Aufgabenstellung, etc.). **Teile solcher Aufgaben können aber durchaus in einer Prüfung vorkommen!**.

Diese Aufgaben sind dazu da, über den Tellerrand hinaus zu schauen und/oder die Theorie in einen grösseren Kontext zu stellen.

Lösung zu E1 ex-zahlmengen-und-intervalle-akrobatik

a) Falsch. $R_0^- = \mathbb{R} \setminus \mathbb{R}^+$ oder $\mathbb{R}^- = \mathbb{R} \setminus \mathbb{R}_0^+$ wäre wahr.

b) Wahr.

c) Wahr.

d) Falsch. $\mathbb{Q}_0^+ \cap \mathbb{Z} = \mathbb{N}$ oder $\mathbb{Q}^+ \cap \mathbb{Z} = \mathbb{N}^+$ wäre wahr.

e) Wahr.

f) Falsch. Hier ist nicht so klar, wie man es korrigieren kann. Man könnte natürlich $=$ durch $\neq$ ersetzen; oder $\mathbb{Q}$ durch $\varnothing$.

Lösung zu E2 ex-funktionswerte-berechnen

a) $a(2) = 2 + 5 = 7$

b) $m(7) = 3 \cdot 7 = 21$

c) $k(5) = \frac{1}{5}$

d) $q(-1) = (-1)^2 = 1$

e) $a\left(\frac{2}{3}\right) = \frac{2}{3} + 5 = \frac{17}{3}$

f) $m\left(-\frac{3}{4}\right) = 3 \cdot \left(-\frac{3}{4}\right) = -\frac{9}{4}$

g) $k\left(\sqrt{2}\right) = \frac{1}{\sqrt{2}}$

h) $a(x-2) = x - 2 + 5 = x + 3$

i) $m(u+v) = 3 \cdot (u+v) = 3u + 3v$

j) $q(c+d) = (c+d)^2 = c^2 + 2cd + d^2$

k) $a(m(4)) = a(3 \cdot 4) = a(12) = 12 + 5 = 17$

l) $m(a(4)) = m(4+5) = m(9) = 3 \cdot 9 = 27$

m) $q(m(k(-0.5))) = (3 \cdot \frac{1}{-0.5})^2 = 36$

n) $a(k(q(\sqrt{7}))) = \frac{1}{(\sqrt{7})^2} + 5 = \frac{36}{7}$

Lösung zu E3 ex-funktionen-notieren

(a)

$$\begin{aligned} w\colon \mathbb{R}_0^+ &\to \mathbb{R}, \\ x &\mapsto \sqrt[3]{x} \end{aligned}$$

(b)

$$\begin{aligned} d\colon \mathbb{N} &\to \mathbb{Z}, \\ x &\mapsto -2x \end{aligned}$$

(c)

$$\begin{aligned} n\colon [0,20] &\to [1,6], \\ p &\mapsto \frac{p}{20} \cdot 5 + 1 = 1 + 5 \cdot \frac{p}{20} \end{aligned}$$

Die Funktion n wird oft beim Berechnen von Schulnoten verwendet: n steht für «Note», p für «Punktzahl».

Lösung zu E4 ex-funktionen-R2-R2

(a) Punktspiegelung p am Ursprung = Streckung am Ursprung mit Streckfaktor -1.

$$p\colon \mathbb{R}^2 \to \mathbb{R}^2,\quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x \\ -y \end{pmatrix}$$

(b) Verschiebung v um 2 Einheiten nach rechts und 3 Einheiten nach oben.

$$v\colon \mathbb{R}^2 \to \mathbb{R}^2,\quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+2 \\ y+3 \end{pmatrix}$$

(c) Reflektion/Spiegelung r an der x-Achse.

$$r\colon \mathbb{R}^2 \to \mathbb{R}^2,\quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y \end{pmatrix}$$

(d) Spiegelung r' an der y-Achse.

$$r'\colon \mathbb{R}^2 \to \mathbb{R}^2,\quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x \\ y \end{pmatrix}$$

(e) Spiegelung r'' an der «ersten Winkelhalbierenden» (= der Winkelhalbierenden der beiden Koordinateachsen, die durch den Punkt $(1, 1)$ geht).

$$r''\colon \mathbb{R}^2 \to \mathbb{R}^2,\quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}$$

(f) Drehung d um den Ursprung des Koordinatensystems um 90° in mathematisch positiver Drehrichtung.

$$d\colon \mathbb{R}^2 \to \mathbb{R}^2,\quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ x \end{pmatrix}$$

Lösung zu E5 ex-well-defined-function

The definition of f is wrong because the $\sqrt{x}$ is only defined if $x \geq 0$ but the domain in the definition is the set of all real numbers.

The definition of g is wrong because $\sqrt{y-1}$ is only defined if $y - 1 \geq 0$ or equivalently $y \geq 1$. Moreover, the codomain should be $\mathbb{R}$ or $\mathbb{R}_0^+$, but not $\mathbb{Z}$.

The definition of h is wrong because $z + 1$ must be non-zero, so we need to require $z \neq -1$.

Correct definitions are:

$$f\colon \mathbb{R}_0^+ \to \mathbb{R},\ x \mapsto \sqrt{x} \qquad g\colon [1, \infty) \to \mathbb{R},\ y \mapsto \sqrt{y-1} \qquad h\colon \mathbb{R} \setminus \{-1\} \to \mathbb{R},\ z \mapsto \tfrac{1}{z+1}$$

In the first two examples, $\mathbb{R}_0^+$ would also be a valid codomain (as would be every bigger subset of $\mathbb{R}$).
In the third example, $\mathbb{R} \setminus \{0\}$ would also be a valid codomain.

Lösung zu E6 ex-graph-trogener-bahn

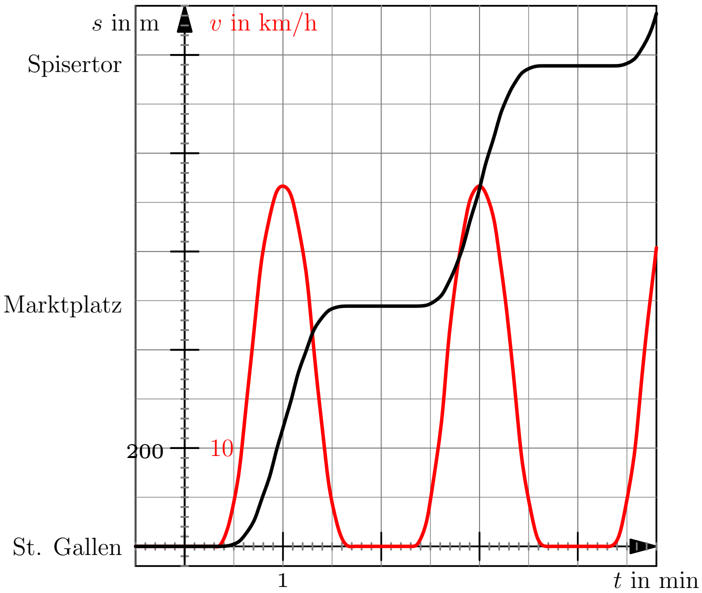
Unten sind zwei Zeichnungen angegeben. Die rechte Zeichnung ist im Wesentlichen richtig, jedoch kann die Bahn hier aus dem Stillstand augenblicklich mit konstanter Geschwindigkeit losfahren und ähnlich augenblicklich die Geschwindigkeit auf Null reduzieren. (Dass die rote und die schwarze Kurve teilweise auf derselben Höhe verlaufen, ist Zufall.)

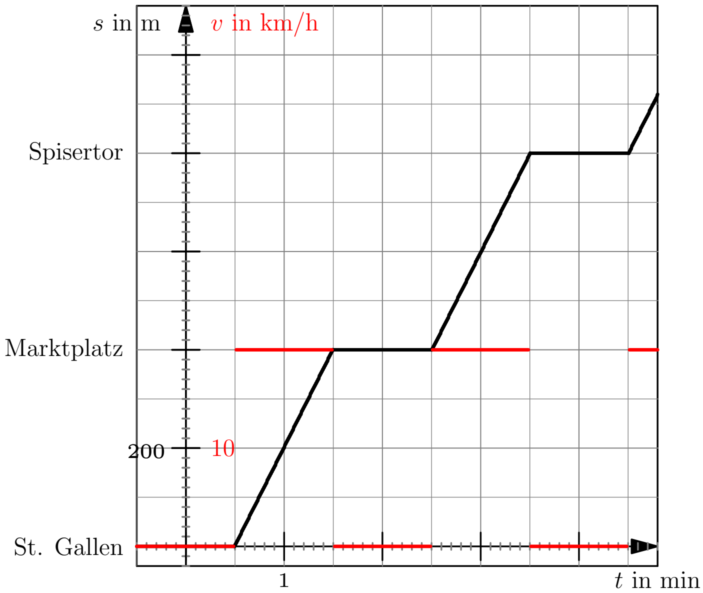
Die linke Zeichnung ist realistischer: Die Geschwindigkeit steigt beim Losfahren und sinkt dann wieder beim Bremsen/Ausrollen.

(a) Die schwarze Kurve zeigt die zurückgelegte Strecke in Abhängigkeit von der Zeit.

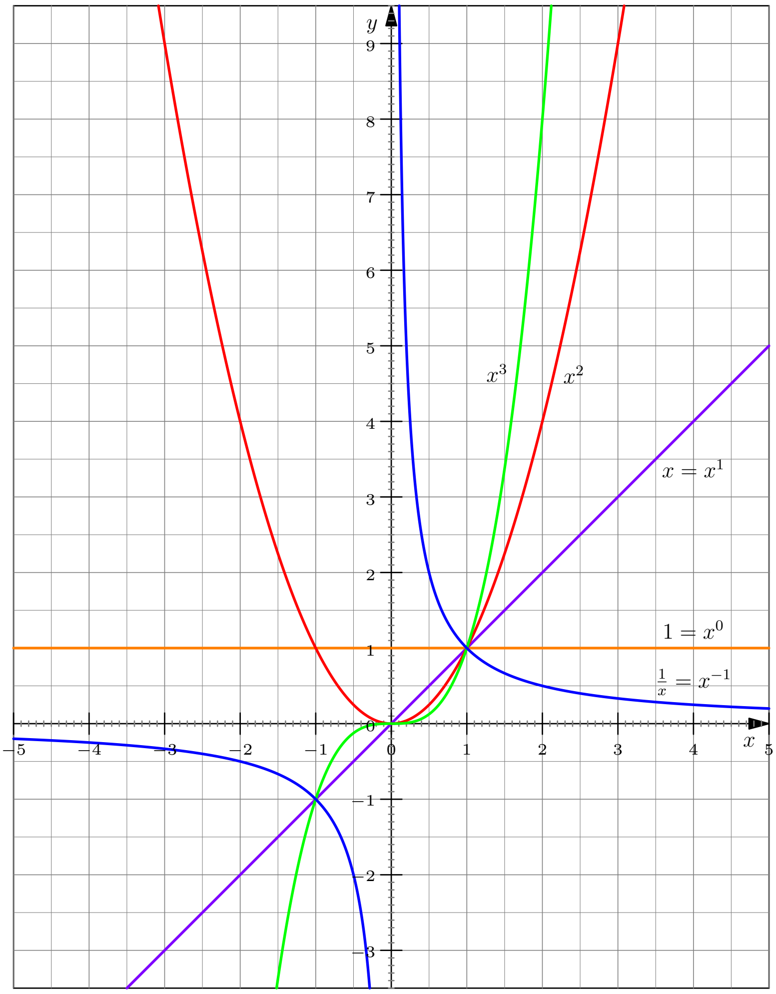
(b) Die rote Kurve zeigt die aktuelle Geschwindigkeit in Abhängigkeit von der Zeit.

(c) Die rote Kurve (= Geschwindigkeit) ist umso höher, je steiler die schwarze Kurve ist.
Genauer Zusammenhang: Die rote Kurve ist die Ableitung (= Steigung der Tangenten) an die schwarze Kurve: Wenn man zu einem gegebenen Zeitpunkt t im Punkt $(t, s(t))$ eine Tangente an die schwarze Kurve legt (also eine Gerade, die die schwarze Kurve möglichst gut approximiert), so ist $v(t)$ die «Steigung» dieser Tangenten. Die Steigung gibt an, um wie viel sich die Tangente in vertikaler Richtung ändert, wenn man eine Einheit nach rechts geht.

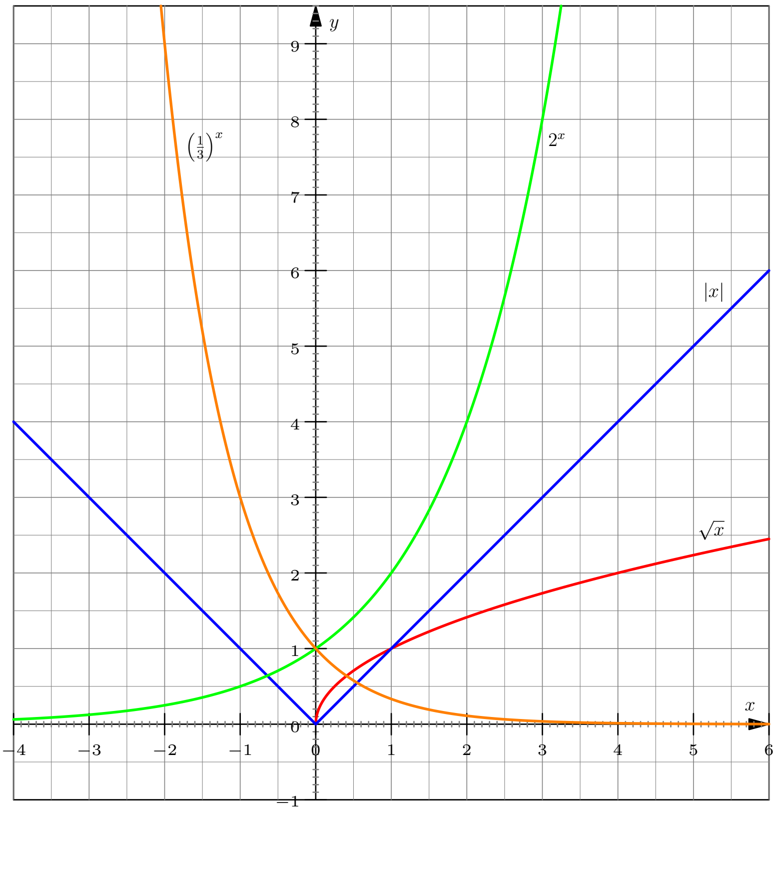




Lösung zu E7 ex-wichtige-graphen-zeichnen



(a)



(b)

Lösung zu E8 ex-punkt-auf-graph

(a) Nein, denn $f(3) = 9 - 6 - 8 = -5 \neq -4$
(b) Ja, denn $f(-3) = 9 + 6 - 8 = 7$
(c) Für $y = f(2) = 4 - 4 - 8 = -8$
(d) Im Punkt $(0, f(0)) = (0, -8)$.
(e) Die Gleichung $f(x) = 16$ bzw. ausgeschrieben $x^2 - 2x - 8 = 16$.
Die Lösungen sind übrigens $x = -4$ und $x = 6$ (denn $x^2 - 2x - 24 = (x + 4)(x - 6)$).
(f) Die Gleichung $f(x) = 0$ bzw. ausgeschrieben $x^2 - 2x - 8 = 0$.
Die Lösungen sind übrigens $x = -2$ und $x = 4$ (denn $x^2 - 2x - 8 = (x + 2)(x - 4)$).

Lösung zu E9 ex-funktionsgraph-oder-nicht

- orange: Kein Funktionsgraph, da bei einer Funktion einem Argument nur **ein** Funktionswert zugeordnet wird. Hier würde aber dem Argument -2 sowohl der Wert 1 als auch der Wert -1 zugeordnet. Dasselbe Problem gibt es für jedes Argument x mit $-3 < x < -1$.
- grün: Ist ein Funktionsgraph, die Definitionsmenge ist das Intervall $[-3, 3]$
- rot: Ist kein Funktionsgraph, da allen Argumenten, die echt grösser als 1 sind, zwei Funktionswerte zugeordnet werden müssten.
- blau: Ist ein Funktionsgraph, die Definitionsmenge ist das Intervall $\mathbb{D} = [-2.5, 2]$.
- lila: Ist ein Funktionsgraph, die Definitionsmenge ist $\mathbb{D} = [0, 2]$.
- magentarot: Ist ein Funktionsgraph, die Definitionsmenge ist $\mathbb{D} = [-2.5, -1.5] \cup [-1.2, 0.5]$.
- olivfarben: Ist kein Funktionsgraph, denn dem Argument 1 würden unendlich viele Werte zugewiesen.
- braun: Ist kein Funktionsgraph, denn dem Argument 2.5 würden unendlich viele Werte zugewiesen.

Lösung zu E10 ex-funktionen-ablesen

(a) Die ungefähren Funktionswerte sind:

$f(-4)$	≈ 1.7	$g(-4)$	≈ 1.4	$h(-4)$	≈ -1.5
$f(-3)$	≈ 0	$g(-3)$	≈ 1.2	$h(-3)$	≈ -0.8
$f(-2)$	≈ -1.7	$g(-2)$	≈ 1	$h(-2)$	≈ 0
$f(-1)$	≈ -1.7	$g(-1)$	≈ 0.7	$h(-1)$	≈ 2
$f(0)$	≈ 0	$g(0)$	≈ 0.4	$h(0)$	≈ 0
$f(1)$	≈ 1.7	$g(1)$	≈ 0	$h(1)$	≈ -0.8
$f(2)$	≈ 1.7	$g(2)$	≈ -1	$h(2)$	≈ -1.5
$f(3)$	≈ 0	$g(3)$	nicht definiert	$h(3)$	≈ -2
$f(4)$	≈ -1.7	$g(4)$	nicht definiert	$h(4)$	≈ -2.5

(b) Man kann natürlich nur die Nullstellen im angezeigten Bereich angeben.
Die Nullstellen von f sind -3 und 0 und 3.
Die Funktion g hat nur die Nullstelle 1.
Die Funktion h hat die Nullstellen -2 und 0.

Lösung zu E11 ex-maximale-definitionsmenge-bestimmen

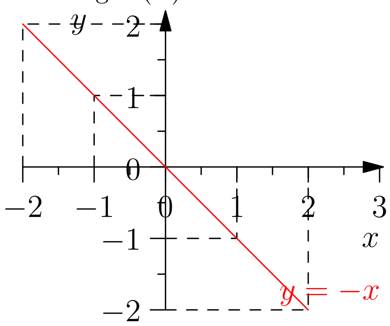
a) $f(x) = \frac{1}{x}$ $\qquad \mathbb{D} = \mathbb{R}^*$
b) $f(x) = \sqrt{x}$ $\qquad \mathbb{D} = \mathbb{R}_0^+$
c) $f(x) = \frac{1}{x-2}$ $\qquad \mathbb{D} = \mathbb{R} \setminus \{2\}$
d) $f(x) = \sqrt{-x}$ $\qquad \mathbb{D} = \mathbb{R}_0^-$
e) $f(x) = \frac{1}{x+5}$ $\qquad \mathbb{D} = \mathbb{R} \setminus \{-5\}$
f) $f(x) = \sqrt{x+5}$ $\qquad \mathbb{D} = [-5, \infty)$
g) $f(x) = \frac{1}{2x-5}$ $\qquad \mathbb{D} = \mathbb{R} \setminus \{\frac{5}{2}\}$
h) $f(x) = \sqrt{x^2+1}$ $\qquad \mathbb{D} = \mathbb{R}$

i) $f(x) = \frac{1}{(x-2)(x+2)}$ $\mathbb{D} = \mathbb{R} \setminus \{-2, +2\}$
j) $f(x) = \sqrt{-x+5}$ $\mathbb{D} = (-\infty, 5]$
k) $f(x) = \frac{1}{x^2-6x+9}$ $\mathbb{D} = \mathbb{R} \setminus \{3\}$
l) $f(x) = \sqrt{16-x^2}$ $\mathbb{D} = [-4, 4]$
m) $f(x) = \frac{1}{x^4+4x^3+6x^2+4x+1}$ $\mathbb{D} = \mathbb{R} \setminus \{-1\}$
n) $f(x) = \sqrt{x^2-16}$ $\mathbb{D} = (-\infty, -4] \cup [4, \infty)$
o) $f(x) = \sqrt{2+x} + \sqrt{2-x}$ $\mathbb{D} = [-2, 2]$
p) $f(x) = \frac{1}{\sqrt{2+x}} + \frac{1}{\sqrt{2-x}}$ $\mathbb{D} = (-2, 2)$

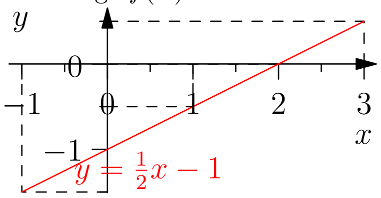
Lösung zu E12 ex-funktionen-zeichnen

(Definitions- und Bildmenge sind in den Zeichnungen noch nicht farblich markiert.)

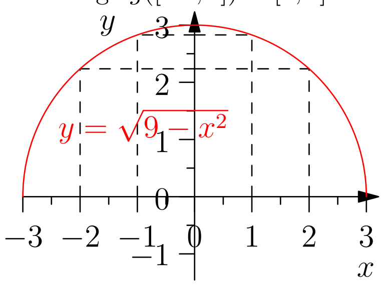
a) Definitionsmenge $\mathbb{D} = \mathbb{R}$,
Bildmenge $e(\mathbb{R}) = \mathbb{R}$



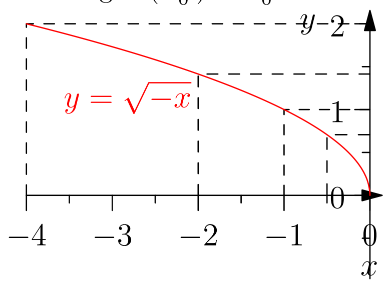
b) Definitionsmenge $\mathbb{D} = \mathbb{R}$,
Bildmenge $f(\mathbb{R}) = \mathbb{R}$



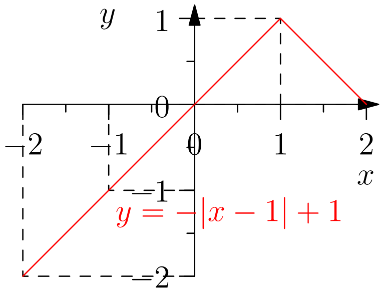
c) Definitionsmenge $\mathbb{D} = [-3, 3]$,
Bildmenge $g([-3, 3]) = [0, 3]$



d) Definitionsmenge $\mathbb{D} = R_0^-$,
Bildmenge $h(\mathbb{R}_0^-) = \mathbb{R}_0^+$



e)
Definitionsmenge $\mathbb{D} = R$,
Bildmenge $i(\mathbb{R}) = (-\infty, 1]$



Lösung zu E13 ex-geraden-verschiedener-steigungen
Die Ursprungsgerade mit Steigung $-\frac{1}{4}$ ist die Gerade durch den Ursprung $(0, 0)$ und den Punkt $(1, -\frac{1}{4})$ (oder äquivalent durch den Ursprung und den Punkt $(4, -1)$). Die anderen gesuchten Ursprungsgeraden konstruiert man analog.

Lösung zu E14 ex-spezielle-steigungen

Hinweis: In der Lösung hier werden nur positive Steigungen berechnet. Spiegelt man die Geraden an der x-Achse bleibt der Winkel gleich, die Steigung wird aber negativ. D. h. zu allen Aufgaben ist auch die entsprechend negative Steigung eine Lösung.

a) Die Steigung ist 0.

b) Man zeichnet ein 30°-60°-90°Stützdreieck, z. B. so, dass die Hypotenuse 1 ist. Dann sind die Katheten $\frac{1}{2}$ (y-Differenz) und $\frac{\sqrt{3}}{2}$ (x-Differenz). Die Steigung ist somit

$$\frac{\Delta y}{\Delta x} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \ldots = \frac{1}{\sqrt{3}} = \frac{1 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{3}}{3}$$

c) Steigung 1.

d) Wie in b), einfach x und y vertauschen, also ist die Steigung $\sqrt{3}$.

e) Die Steigung ist nicht definiert, denn Division durch Null ist nicht definiert. (Genau genommen hätte man diesen Fall in der Definition der Steigung ausschliessen müssen.) Die Steigung wäre quasi ∞.

Lösung zu E15 ex-graphen-zeichnen-und-steigung-und-y-achsenabschnitt-bestimmen

Alle Graphen sind Geraden.

Die Graphen bitte selbst kontrollieren, indem man sie etwa mit GeoGebra oder einer anderen Computer-Algebra-Software (CAS) zeichnet.

Wenn man die Funktion auf die «Standard-Form» $\ell(x) = mx + q$ gebracht hat, ist m die Steigung und q der y-Achsenabschnitt. Den y-Achsenabschnitt linearer Funktionen bekommt man auch durch Einsetzen von $x = 0$.

a) $a(x) = 3x = 3x + 0$ hat Steigung $m = 3$ und y-Achsenabschnitt $q = 0 = a(0)$.

b) $b(x) = \frac{1}{3}x$ hat Steigung $m = \frac{1}{3}$ und y-Achsenabschnitt $q = 0 = b(0)$.

c) $c(x) = -x$ hat Steigung -1 und y-Achsenabschnitt 0.

d) $d(x) = -\frac{1}{2}x$ hat Steigung $-\frac{1}{2}$ und y-Achsenabschnitt 0.

e) $e(x) = x - 1$ hat Steigung 1 und y-Achsenabschnitt $q = -1 = e(0)$.

f) $f(x) = -x + 1$ hat Steigung -1 und y-Achsenabschnitt 1.

g) $g(x) = 2x - 2$ hat Steigung 2 und y-Achsenabschnitt -2.

h) $h(x) = -\frac{1}{3}x + \frac{2}{3}$ hat Steigung $-\frac{1}{3}$ und y-Achsenabschnitt $\frac{2}{3}$.

i) $i(x) = -3 = 0x - 3$ hat Steigung 0 und y-Achsenabschnitt -3.

j) $j(x) = 0 = 0x + 0$ hat Steigung 0 und y-Achsenabschnitt 0.

Lösung zu E16 ex-lineare-funktionen-ablesen

a) y-Achsenabschnitt $q = \frac{1}{2}$. Steigung $m = \frac{1}{2}$ (z. B. mit den Punkten $(-1, 0)$ und $(1, 1)$, $\frac{\Delta y}{\Delta x} = \frac{1}{2}$. Also $f(x) = \frac{1}{2}x + \frac{1}{2}$.

b) y-Achsenabschnitt $q = -\frac{1}{2}$. Steigung $m = \frac{3}{2}$ (z. B. mit den Punkten $(-1, -2)$ und $(1, 1)$, $\frac{\Delta y}{\Delta x} = \frac{3}{2}$. Also $f(x) = \frac{3}{2}x - \frac{1}{2}$.

c) y-Achsenabschnitt $q = 1$. Steigung $m = -\frac{1}{3}$ (z. B. mit den Punkten $(0, 1)$ und $(3, 0)$, $\frac{\Delta y}{\Delta x} = \frac{-1}{3}$. Also $f(x) = -\frac{1}{3}x + 1$.

d) y-Achsenabschnitt $q = -\frac{3}{2}$. Steigung $m = -2$ (z. B. mit den Punkten $(-1, \frac{1}{2})$ und $(1, -\frac{7}{2})$, $\frac{\Delta y}{\Delta x} = \frac{-4}{2}$. Also $f(x) = -2x - \frac{3}{2}$.

Lösung zu E17 ex-gerade-mit-steigung-durch-punkt

(a) Bestimme die Gleichung der Geraden g mit Steigung $m = -\frac{2}{3}$, die durch den Punkt $A = (-1, -3)$ geht.
Allgemeiner Ansatz: $\ell(x) = mx + q$
Da die Steigung m gegeben ist, gilt $y = \ell(x) = -\frac{2}{3}x + q$.
Dass der Punkt $A = (-1, -3)$ auf dem Graphen von ℓ liegt, ist gleichbedeutend zu $\ell(-1) = -3$ bzw. ausgeschrieben

$$-\frac{2}{3} \cdot (-1) + q = -3$$

(Man ersetzt in der obigen suggestiven Schreibweise $y = -\frac{2}{3}x + q$ also x und y durch die entsprechenden Koordinaten des Punktes A.)
Auflösen dieser linearen Gleichung nach q liefert $q = -3 - \frac{2}{3} = -\frac{11}{3}$.
Ergebnis:

$$\ell(x) = -\frac{2}{3}x - \frac{11}{3}$$

- Teste, ob der Punkt A auf dem Graphen deiner Gleichung liegt.
  Ja, denn es gilt $\ell(-1) = -\frac{2}{3} \cdot (-1) - \frac{11}{3} = \frac{2}{3} - \frac{11}{3} = \frac{-9}{3} = -3$. Test erfolgreich.
- Liegt der Punkt $(4, 4)$ auf der Geraden?
  Nein, denn $\ell(4) = -\frac{2}{3} \cdot 4 - \frac{11}{3} = -\frac{8}{3} - \frac{11}{3} = \frac{-19}{3} = -\frac{19}{3} \neq 4$. Der Punkt liegt also nicht auf der Geraden.
  Anschaulich ist das auch klar: Wenn die Gerade durch den Punkt $A = (-1, -3)$ geht und negative Steigung hat, bleibt sie «rechts von A» unterhalb der x-Achse. Der Punkt $(4, 4)$ liegt aber «rechts von A» und oberhalb der x-Achse.
- Für welches y liegt der Punkt $(3, y)$ auf der Geraden?
  Für $y = \ell(3) = -\frac{2}{3} \cdot 3 - \frac{11}{3} = -2 - \frac{11}{3} = \frac{-6-11}{3} = \frac{-17}{3}$.

(b) siehe Lehrerversion des Skripts

Lösung zu E18 ex-gerade-durch-zwei-punkte

(a) Mit konkreten Zahlen: Bestimme die Gleichung der Geraden g, die durch die beiden Punkte $A = (-1, 2)$ und $B = (5, -1)$ geht.
  (i) 1. Lösungsweg:
  Die Steigung ist

$$m = \frac{\Delta y}{\Delta x} = \frac{y_A - y_B}{x_A - x_B} = \frac{2 - (-1)}{-1 - 5} = \frac{3}{-6} = -\frac{1}{2}$$

Nun kennen wir die Steigung der Geraden und zwei Punkte auf der Geraden, d.h. wir können aus Steigung und einem Punkt (kann A oder B wählen) die Geradengleichung wie in Aufgabe E17 bestimmen (dem Leser überlassen).
Ergebnis: $g(x) = -\frac{1}{2}x + \frac{3}{2}$

- Teste, ob die beiden Punkte A und B auf dem Graphen deiner Gleichung liegen.
  Wegen $g(-1) = -\frac{1}{2} \cdot (-1) + \frac{3}{2} = \frac{4}{2} = 2$ liegt $A = (-1, 2)$ auf dem Graphen von ℓ.
  Wegen $g(5) = -\frac{1}{2} \cdot 5 + \frac{3}{2} = \frac{-2}{2} = -1$ liegt $B = (5, -1)$ auf dem Graphen von ℓ.
- 2. Lösungsweg: Die Gerade hat die Gleichung $g(x) = mx + q$. Zu bestimmen sind m und q.
  - $A \in g$ ist gleichbedeutend zu $g(-1) = 2$, d.h. $m \cdot (-1) + q = 2$
  - $B \in g$ ist gleichbedeutend zu $g(5) = -1$, d.h. $m \cdot 5 + q = -1$

  Also müssen diese beiden jeweils letztgenannten Gleichungen gelten (die beiden Gleichungen bilden ein Gleichungssystem in den beiden Variablen m und q). Löst man beide nach q auf, so erhält man

$$q = 2 + m$$
$$q = -1 - 5m$$

  Also gilt $2 + m = -1 - 5m$. Auflösen nach m ergibt $m = -\frac{1}{2}$.
  Setzt man dies in einer der beiden obigen Gleichungen für q ein, so erhält man $q = 2 + (-\frac{1}{2}) = -1 - 5 \cdot (-\frac{1}{2}) = \frac{3}{2}$.
  Die gesuchte Geradengleichung ist also $g(x) = -\frac{1}{2}x + \frac{3}{2}$.

(b) siehe Lehrerversion des Skripts

Lösung zu E19 ex-schnittpunkt

Wenn x die x-Koordinate des Schnittpunktes ist, gilt

$$\begin{aligned} f(x) &= g(x) \\ -\frac{1}{3}x - 1 &= \frac{3}{2}x + 1 && |\cdot 6 \\ -2x - 6 &= 9x + 6 && |+2x-6 \\ -12 &= 11x && |:11 \\ -\frac{12}{11} &= x \end{aligned}$$

Eingesetzt in eine der Funktionen: $f\left(-\frac{12}{11}\right) = \frac{12}{33} - 1 = -\frac{7}{11}$. Zur Kontrolle (nicht wirklich nötig) eingesetzt in g: $g\left(-\frac{12}{11}\right) = -\frac{36}{22} + 1 = -\frac{7}{11}$.
Und damit sind die Koordinaten des Schnittpunktes: $\left(-\frac{12}{11}, -\frac{7}{11}\right)$

Lösung zu E20 ex-senkrechte-geraden-steigung

(a) Man zeichne die Gerade g und ergänze ein Steigungsdreieck (etwa mit $\Delta x = 5$ und $\Delta y = 3$).
Die Gerade h entsteht aus g durch eine Drehung um den Ursprung um den Winkel 90°. Dreht man auch das Steigungsdreieck auf diese Weise, so erhält man ein Steigungsdreieck zu h. Sein «Δx ist -3, das Negative des vorherigen Δy», sein «Δy ist 5, das vorherige Δx». Damit hat h die Steigung $\frac{5}{-3} = -\frac{5}{3}$.

(b) Steigung $-\frac{5}{3}$.
Der Unterschied zur vorherigen Teilaufgabe ist nur, dass nun beliebige Geraden statt Ursprungsgeraden betrachtet werden. Jede Gerade hat aber eine dazu parallelel Ursprungsgerade mit derselben Steigung.

(c) siehe Lehrerversion des Skripts

Lösung zu E21 ex-zug-begegnung

(a) (zu ergänzen)

(b) Wir rechnen in Kilometer (als Längeneinheit) ab Wattwil und Stunden (als Zeiteinheit) ab 9:00 Uhr. Es bezeichnet t die Zeit.
- Schnellzug: $s(t) = 90\left(t - \frac{2}{60}\right) = 90t - 3$
- Regionalzug: $r(t) = 8 - 60t$.

Zur Begegnungszeit muss $s(t) = r(t)$ gelten, also

$$90t - 3 = 8 - 60t$$

Auflösen der Gleichung nach t ergibt $t = \frac{11}{150}$.
Die Begegnung findet also um 9:04:24 Uhr statt.
Wer mag, kann auch berechnen, dass die Begegnung bei Kilometer 3.6 ab Wattwil stattfindet.

Lösung zu E22 ex-bildschirm

(a)
- $U = (0, 0)$ ist im s-z-Koordinatensystem $U = (480, 720)$;
- $A = (12, 6)$ ist im s-z-Koordinatensystem $A = (1920, 0)$;
- $B = (-4, -3)$ im s-z-Koordinatensystem ist $B = (0, 1080)$.

(b) Sei $s(x)$ die lineare Umrechnungsfunktion, die aus der x-Koordinate x eines Punktes seine s-Koordinate (= Spaltenkoordinate) berechnet. Zum Beispiel gilt $s(12) = 1920$.
- $s(0) = 480$; $s(-4) = 0$
- (zu ergänzen)
- $s(x) = \frac{1920-480}{12}x + 480 = 120x + 480$
- Test: Die Funktion ist linear und es gelten $s(12) = 120\cdot 12 + 480 = 1920$ und $s(-4) = 120\cdot(-4) + 480 = 0$ wie gewünscht.

(c) $z(y) = \frac{-720}{6}y + 720 = -120y + 720$
Tests: $z(-3) = -120 \cdot (-3) + 720 = 1080$, $z(0) = 720$ und $z(6) = -120 \cdot 6 + 720 = 0$

Lösung zu E23 ex-abstand-paralleler-geraden

(a) dem Leser überlassen
(b) dem Leser überlassen
(c) Eine Möglichkeit besteht darin, die Konstruktion rechnerisch nachzuvollziehen. Dazu schneidet man eine Gerade, die auf den beiden (parallelen) Geraden senkrecht steht, mit diesen beiden Geraden; als senkrechte Gerade kann man zum Beispiel die Gerade mit der Funktionsgleichung $k(x) = -\frac{1}{2}x$ nehmen.
Man löst die Gleichungen $f(x) = k(x)$ und $g(x) = k(x)$ und erhält als Schnittpunkt mit f den Punkt $\left(\frac{2}{5}, -\frac{1}{5}\right)$ und als Schnittpunkt mit g den Punkt $\left(-\frac{2}{5}, \frac{1}{5}\right)$. Der Abstand der beiden Punkte beträgt $\sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \sqrt{\frac{20}{25}} = \sqrt{\frac{2^2 \cdot 5}{5^2}} = \frac{2}{5}\sqrt{5} \approx 0.89443$
(d) Seien $f(x) = mx$ und $g(x) = mx - q$. Die Gleichung einer Rechtwinkligen Gerade ist $k(x) = -\frac{1}{m}x$. Der Schnittpunkt mit f ist $(0, 0)$, der Schnittpunkt mit g ist $\left(\frac{q}{m+\frac{1}{m}}, -\frac{q}{m^2+1}\right)$. Der Abstand der beiden Punkte ist also $\sqrt{\left(\frac{q}{m+\frac{1}{m}}\right)^2 + \left(\frac{q}{m^2+1}\right)^2} = \sqrt{\left(\frac{qm}{m^2+1}\right)^2 + \left(\frac{q}{m^2+1}\right)^2} = \sqrt{\frac{q^2(m^2+1)}{(m^2+1)^2}} = \frac{q\sqrt{m^2+1}}{m^2+1} = \frac{q}{\sqrt{1+m^2}}$

Lösung zu E24 ex-geraden-durch-punkte

(a) Steigungen $m = \frac{\Delta y}{\Delta x}$: $m_a = \frac{3}{2}$, $m_b = 0$, $m_c = -1$.
Achsenabschnitte: Ein Punkt auf der Geraden in die Funktionsgleichung mit unbekanntem q einsetzen, nach q auflösen. Beispiel für die Gerade a:

$$\begin{aligned} f_a(1) &= -2 \\ m_a \cdot 1 + q_a &= -2 \\ \frac{3}{2} + q_a &= -2 && \left| -\frac{3}{2} \right. \\ q_a &= -\frac{7}{2} \end{aligned}$$

Entsprechend $q_b = 1$, $q_c = -1$ und damit

$$f_a(x) = \frac{3}{2}x - \frac{7}{2} \qquad f_b(x) = 1 \qquad f_c(x) = -x - 1$$

(b) Nein, da $m_a \neq -\frac{1}{m_c}$
(c) $f_h(x) = m_h \cdot x + q_h$ mit $m_h = -\frac{1}{\frac{3}{2}} = -\frac{2}{3}$. q_h erhält man durch Einsetzen der Koordinaten von A und Auflösen nach q_h. Resultat: $f_h(x) = -\frac{2}{3} \cdot x - \frac{1}{3}$.
(d) Auflösen der Gleichung $f_a(x) = f_h(x) \Leftrightarrow \frac{3}{2}x - \frac{7}{2} = -\frac{2}{3}x - \frac{1}{3}$ liefert $x = \frac{19}{13}$. Eingesetzt erhält man $y = f_a\left(\frac{19}{13}\right) = \frac{3}{2} \cdot \frac{19}{13} - \frac{7}{2} = -\frac{34}{26} = -\frac{17}{13}$

Lösung zu E25 ex-weg-a-fluss-b
Dörfer $A = (1, 2)$ und $B = (5, 3)$.

(a) Spiegelt man $B = (5, 3)$ am Fluss (= der x-Achse), so erhält man den Punkt $B' = (5, -3)$.
Der kürzeste Weg von Dorf A nach Dorf B, bei dem man unterwegs den Fluss besucht, ist dann genauso lang wie der kürzeste Weg von A nach B' (Skizze erstellen, wenn das nicht klar ist).
Also hat dieser Weg die Länge

$$\sqrt{(5-1)^2 + (-3-2)^2} = \sqrt{16+25} = \sqrt{41}$$

(b) Die Gerade g durch A und B' schneidet die x-Achse in dem Punkt, in dem man den Fluss auf dem kürzesten Weg besucht.

Die Gleichung von g ist $g(x) = -\frac{5}{4}x + \frac{13}{4}$.
Ihren Schnittpunkt S mit der x-Achse erhält man durch Lösen der Gleichung $g(x) = 0$, d.h. $-\frac{5}{4}x + \frac{13}{4} = 0$. Man erhält die Lösung $x = \frac{13}{5}$.
Der Schnittpunkt S ist also $\left(\frac{13}{5}, 0\right)$.

Lösung zu E26 ex-rechtwinkliges-dreieck-konstruieren

(a) selbst

(b) Vorgehen: Zuerst werden die Funktionsgleichungen der Geraden b und $a = BC$ bestimmt. Ihr Schnittpunkt ist dann der gesuchte Punkt c.
Geradengleichung von b (Steigung $\frac{3}{2}$): $b(x) = \frac{3}{2}x + q_b$. Es gilt $A \in b$: Setzt man die x-Koordinate von A in die Funktion b ein, erhält man die y-Koordinate von A:

$$\begin{aligned} b(-2) &= -1 \\ \frac{3}{2} \cdot -2 + q_b &= -1 \\ -3 + q_b &= -1 && \mid +3 \\ q_b &= 2 \end{aligned}$$

Also gilt $b(x) = \frac{3}{2}x + 2$.
Die Gerade a ist senkrechts auf/rechtwinklig zu b, hat also die Steigung $-\frac{1}{\frac{3}{2}} = -\frac{2}{3}$. Es gilt $B \in a$, also

$$\begin{aligned} a(3) &= 1 \\ -\frac{2}{3} \cdot 3 + q_a &= 1 \\ -2 + q_a &= 1 && \mid +2 \\ q_a &= 3 \end{aligned}$$

Also gilt $a(x) = -\frac{2}{3}x + 3$.
Der gesuchte Punkt C ist der Schnittpunkt der Geraden $a(x) = -\frac{2}{3}x + 3$ und $b(x) = \frac{3}{2}x + 2$.

$$\begin{aligned} -\frac{2}{3}x + 3 &= \frac{3}{2}x + 2 && \mid -2 + \frac{2}{3}x \\ 1 &= \frac{13}{6}x && \mid \cdot \frac{6}{13} \\ \frac{6}{13} &= x \end{aligned}$$

Die y-Koordinate des Schnittpunktes erhält man durch Einsetzen (in a oder b):

$$a\left(\frac{6}{13}\right) = -\frac{2}{3} \cdot \frac{6}{13} + 3 = \frac{35}{13}$$

Kontrolle (eigentlich unnötig):

$$b\left(\frac{6}{13}\right) = \frac{3}{2} \cdot \frac{6}{13} + 2 = \frac{35}{13}$$

Damit sind die Koordinaten von $C = \left(\frac{6}{13}, \frac{35}{13}\right) \approx (0.4615, 2.6923)$.

Lösung zu E27 ex-hoehe-mittelsenkrechte-seitenhalbierende-winkelhalbierende-berechnen

Eckpunkte des Dreiecks: $A = (2, 1)$, $B = (17, 4)$, $C = (8, 13)$.

(a) Gerade c:

$$c(x) = mx + q = \frac{4-1}{17-2}x + q = \frac{3}{15}x + q = \frac{1}{5}x + q$$

Da A auf c liegt, muss $c(2) = 1$ gelten, also

$$\begin{aligned} \frac{1}{5} \cdot 2 + q &= 1 \\ q &= 1 - \frac{2}{5} = \frac{3}{5} \end{aligned}$$

Also

$$c(x) = \frac{1}{5}x + \frac{3}{5}$$

(b) Höhe h_c. Diese Höhe steht senkrecht auf c, hat also die Steigung $-\frac{1}{\frac{1}{5}} = -5$.

$$h_c(x) = mx + q = -5x + q$$

Da C auf h_c liegt, muss $h_c(8) = 13$ gelten, also

$$\begin{aligned} -5 \cdot 8 + q &= 13 \\ q &= 53 \end{aligned}$$

Also

$$h_c(x) = -5x + 53$$

(c) Die beiden Koordinaten von M_{AB} sind genau die Durchschnitte der jeweiligen Koordinaten von A und B, also

$$M_{AB} = \left(\frac{2+17}{2}, \frac{1+4}{2} \right) = (9.5, 2.5)$$

(d) Mittelsenkrechten m_c. Sie verläuft durch M_{AB} und steht senkrecht auf c, hat also die Steigung -5.

$$m_c(x) = -5x + q$$

Da M_{AB} auf m_c liegt, muss $m_c(9.5) = 2.5$ gelten, also

$$\begin{aligned} -5 \cdot 9.5 + q &= 2.5 \\ -47.5 + q &= 2.5 \\ q &= 50 \end{aligned}$$

Also

$$m_c(x) = -5x + 50$$

(e) Die Seitenhalbierende s_c verläuft durch M_{AB} und C.

$$s_c(x) = \frac{y_C - y_{M_{AB}}}{x_C - x_{M_{AB}}}x + q = \frac{13 - 2.5}{8 - 9.5}x + q = \frac{10.5}{-1.5}x + q = -\frac{21}{3}x + q = -7x + q$$

Da C auf s_c liegt, muss $s_c(8) = 13$ gelten, also

$$\begin{aligned} -7 \cdot 8 + q &= 13 \\ -56 + q &= 13 \\ q &= 69 \end{aligned}$$

Also

$$s_c(x) = -7x + 69$$

(f) Winkelhalbierende w_γ.
Nach dem Satz über die Winkelhalbierende teilt die Winkelhalbierende w_γ die Strecke $[AB]$ im Verhältnis

$$v := \frac{\overline{AC}}{\overline{BC}} = \frac{\sqrt{6^2+12^2}}{\sqrt{9^2+9^2}} = \frac{\sqrt{6^2+4\cdot 6^2}}{\sqrt{2\cdot 9^2}} = \frac{\sqrt{5\cdot 6^2}}{\sqrt{2\cdot 9^2}} = \frac{6\sqrt{5}}{9\sqrt{2}} = \frac{2\sqrt{5}}{3\sqrt{2}} = \frac{2\sqrt{10}}{6} = \frac{\sqrt{10}}{3}$$

Also erhält man den Schnittpunkt W_γ von w_γ mit c, indem man zu A das $\frac{\sqrt{10}}{\sqrt{10}+3}$-fache der Strecke (oder besser des Vektors) $\overrightarrow{AB}$ hinzuaddiert. (Beachte, dass $\frac{\sqrt{10}}{\sqrt{10}+3} = \frac{\sqrt{10}(\sqrt{10}-3)}{(\sqrt{10}+3)(\sqrt{10}-3)} = \frac{10-3\sqrt{10}}{10-9} = 10-3\sqrt{10}$.)

$$\begin{aligned} W_\gamma &= A + (10-3\sqrt{10})\cdot\overrightarrow{AB} \\ &= \begin{pmatrix}2\\1\end{pmatrix} + (10-3\sqrt{10})\cdot\begin{pmatrix}15\\3\end{pmatrix} \\ &= \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}150-45\sqrt{10}\\30-9\sqrt{10}\end{pmatrix} \\ &= \begin{pmatrix}152-45\sqrt{10}\\31-9\sqrt{10}\end{pmatrix} \end{aligned}$$

Die gesuchte Winkelhalbierende w_γ ist die Gerade durch C und W_γ. Die Steigung dieser Geraden ist

$$\begin{aligned} \frac{y_{W_\gamma}-y_C}{x_{W_\gamma}-x_C}x &= \frac{31-9\sqrt{10}-13}{152-45\sqrt{10}-8} \\ &= \frac{18-9\sqrt{10}}{144-45\sqrt{10}} \\ &= \frac{6-3\sqrt{10}}{48-15\sqrt{10}} \\ &= \frac{2-\sqrt{10}}{16-5\sqrt{10}} \\ &= \frac{(2-\sqrt{10})(16+5\sqrt{10})}{(16-5\sqrt{10})(16+5\sqrt{10})} \\ &= \frac{32+10\sqrt{10}-16\sqrt{10}-50}{256-250} \\ &= \frac{-18-6\sqrt{10}}{6} \\ &= -3-\sqrt{10} \end{aligned}$$

Also hat w_γ die Gleichung

$$w_\gamma(x) = (-3-\sqrt{10})x + q$$

Da C auf w_γ liegt, muss $w_\gamma(8) = 13$ gelten, also

$$\begin{aligned} (-3-\sqrt{10})\cdot 8 + q &= 13 \\ q &= 13 + (3+\sqrt{10})\cdot 8 \\ q &= 13 + 24 + 8\sqrt{10} \\ q &= 37 + 8\sqrt{10} \end{aligned}$$

Also hat w_γ die Gleichung

$$w_\gamma(x) = -(3+\sqrt{10})x + 37 + 8\sqrt{10}$$

Lösung zu E28 ex-intervalle-abbilden

(a) $f(x) = -\frac{1}{3}x + \frac{10}{3}$
(b) $f(x) = 2x + 2$ und $f(x) = -2x + 2$
(c) $f(x) = 5x + 1$ und $f(x) = -5x + 6$
(d) $f(x) = \frac{1}{5}x - \frac{1}{5} = \frac{1}{5}(x - 1)$ und $f(x) = -\frac{1}{5}x + \frac{6}{5} = -\frac{1}{5}(6 - x)$
(e) $f(x) = (b - a) \cdot x + a$ und $f(x) = (a - b) \cdot x + b$
(f) $f(x) = \frac{x-a}{b-a} = \frac{1}{b-a} \cdot x - \frac{a}{b-a}$ und $f(x) = \frac{x-b}{a-b} = -\frac{1}{b-a} \cdot x + \frac{b}{b-a}$
(g) $f(x) = \frac{x-a}{b-a} \cdot (d - c) + c$ und $f(x) = \frac{b-x}{b-a} \cdot (d - c) + c$

Teilaufgabe (g) kann wie folgt gelöst werden bzw. das Ergebnis kann wie folgt verstanden werden:

(1) Intervall verschieben: Definiere $f_1(x) = x - a$. Diese Funktion bildet das Intervall $[a, b]$ auf das Intervall $[0, b - a]$ ab.
(2) Das verschobene Intervall $[0, b - a]$ verkleinert man auf das Intervall $[0, 1]$, indem man durch seine Länge dividiert, also $f_2(x) = \frac{f_1(x)}{b-a} = \frac{x-a}{b-a}$. Damit bildet man das ursprüngliche Intervall auf das Intervall $[0, 1]$ ab.
(3) Man vergrössert das Intervall $[0, 1]$ auf die endgültige Länge durch Multiplikation mit $(d-c)$, also $f_3(x) = \frac{x-a}{b-a} \cdot (d - c)$. Damit bildet man das ursprüngliche Intervall auf das Intervall $[0, d - c]$ ab.
(4) Am Schluss verschiebt man durch Addition von c an die endgültige Lage: $f_4(x) = f(x) = \frac{x-a}{b-a} \cdot (d-c)+c$.

Die zweite Lösung erhält man auf nahezu dieselbe Weise: Im ersten Schritt verwendet man die Funktion $f_1(x) = b - x$. Dabei wird pe «$-x$» das Intervall $[a, b]$ «umgedreht» auf das Intervall $[-b, -a]$ abgebildet, dann wird letzteres per «$+b$» auf $[0, b - a]$ abgebildet.

Lösung zu E29 ex-lineare-funktionen-alltagsbeispiele

Einige Beispiele:

- Wenn eine Tonne Weizen 328 Franken kostet, so kosten x Tonnen Weizen $f(x) = 328x$ Franken.
- Ist p der Kilopreis einer Ware in Franken, so kosten x Kilogramm dieser Ware $f(x) = px$ Franken.
- Beträgt die Anmelde-Gebühr in einem Club 100 Franken und die monatliche Gebühr 30 Franken, so zahlt man für die ersten x Monate Mitgliedschaft insgesamt $k(x) = 30x + 100$ Franken.
- Fährt ein Velofahrer mit einer Geschwindigkeit v in $\frac{\text{km}}{\text{h}}$, so legt er in t Stunden eine Strecke von $s(t) = vt$ Kilometern zurück.
- Ein gängige Formel zur Notenberechnung lautet $n(p) = 1 + 5\frac{p}{m} = \frac{5}{m}p + 1$, wobei m die maximal erreichbare Punktzahl ist und p die erreichte Punktzahl.

Lösung zu E30 ex-gerade-transformieren

(a) Spiegelung an x-Achse (Funktionsgleichung mit (-1) multiplizieren):
$f(x) = -mx - q$
(b) Spiegelung an y-Achse (in Funktionsgleichung x durch $-x$ ersetzen):
$g(x) = m(-x) + q = -mx + q$
(c) Spiegelung an der ersten Winkelhalbierenden (geht nur, falls $m \neq 0$):
Man überlegt sich mit einer Skizze, dass die gespiegelte Gerade die Steigung $\frac{1}{m}$ hat (denn in einem Steigungsdreieck werden beim Spiegeln Δx und Δy vertauscht).
Also hat die gespiegelte Gerade die Gleichung $h(x) = \frac{1}{m}x + p$ für ein geeignetes $p \in \mathbb{R}$.
Da $(0, q)$ auf der Ausgangsgeraden liegt, liegt $(q, 0)$ auf der gespiegelten Geraden.
Also muss $h(q) = 0$ gelten oder ausgeschrieben $\frac{1}{m} \cdot q + p = 0$, d.h. $p = -\frac{q}{m}$.
Fazit: $h(x) = \frac{1}{m}x - \frac{q}{m}$
(d) Drehung um 90°.
Die Steigung der gedrehten Geraden ist $-\frac{1}{m}$ (siehe Merke 6.5.19).
Also gilt $i(x) = -\frac{1}{m}x + p$ für ein geeignetes $p \in \mathbb{R}$.
Da $(0, q)$ auf der Ausgangsgeraden liegt, liegt $(-q, 0)$ auf der gedrehten Geraden.
Also muss $i(-q) = 0$ gelten oder ausgeschrieben $-\frac{1}{m} \cdot (-q) + p = 0$, d.h. $p = -\frac{q}{m}$.
Fazit: $i(x) = -\frac{1}{m}x - \frac{q}{m}$

Lösung zu E31 ex-wahr-oder-falsch

(a) **Falsch**. Das gilt nicht wenn die Punkte vertikal übereinander liegen (gleiche x-Koordinate, unterschiedliche y-Koordinaten). Um die Aussage wahr zu machen, könnte man «…Punkte mit unterschiedlichen x-Koordinaten…» schreiben.

(b) **Wahr**. Der einzige Streitpunkt hier ist, ob man identische (übereinander liegende) Geraden ebenfalls als parallel bezeichnet.

(c) **Falsch**. Sie schneiden sich auf der y-Achse.

(d) **Falsch**. Die neue Funktion ist einfach $g(x) = -f(x)$, also an der x-Achse gespiegelt.

(e) **Falsch**. Die Steigung ist Null.

(f) **Wahr**. 1 y-Einheit pro x-Einheit.

(g) **Falsch**. Vertikale Geraden haben keine definierte Steigung (wäre quasi unendlich).

(h) **Wahr**. $m \cdot -\frac{1}{m} = -1$.

(i) **Falsch**. Richtig wäre z. B. «Erhöht man den y-Achsenabschnitt um 2…».

(j) **Wahr**. $h(x) = 2 \cdot f(x) = 2 \cdot (mx + q) = 2m \cdot x + 2q$.

(k) **Wahr.** $f(x) = m_f x + q_f$, $g(x) = m_g x + q_g$, also $h(x) = (m_f + m_g)x + (q_f + q_g)$.

(l) **Wahr.** Wie in (k) erhält man $h(x) = f(m_g x + q_g) = m_f \cdot (m_x x + q_g) + q_f = m_f m_g \cdot x + (m_f q_g + q_f)$.

(m) **Falsch**. Z. B. für $f(x) = x$ und $g(x) = x$ ist $h(x) = x^2$ nicht linear.

(n) **Wahr**. Überprüfen durch einsetzen der x-Koordinaten in die Funktionen.

(o) **Wahr**. Die Gleichung $f(x) = g(x)$ vereinfacht sich auf eine lineare Gleichung (x^2 fällt weg) und die hat genau eine Lösung (der Koeffizient von x ist nicht Null).

(p) **Falsch**. Man kann z. B. die Graphen zeichnen, oder die Gleichung $f(x) = g(x)$ umformen, um $x^2 = -1$ zu finden, was keine Lösung hat.

(q) **Falsch**. Z. B. $f(x) = x^2$ und $g(x) = x - x^2$. Die Summe ist $h(x) = x$ linear.

(r) **Falsch**. Das Produkt genau dann ein lineare Funktion, wenn in mindestens einer Funktion die Steigung gleich Null ist (und damit der quadratische Term weg fällt).

(s) **Falsch**. Das ist nur wahr, wenn die Geraden horizontal sind.

(t) **Falsch**. Der Punkt $(x, f(x))$ liegt auf dem Graphen. Man könnte auch noch monieren, dass x aus dem Definitionsbereich kommen muss.

(u) **Falsch**. Die Notenfunktion ist linear für Punktzahlen zwischen der Punktzahl 0 und der für die Note 6 benötigten Punktzahl. Dort macht der Graph einen Knick und wird horizontal, denn Noten über 6 werden nicht vergeben. Die Funktion setzt sich genaugenommen aus drei linearen Funktionen zusammen. Sind beispielsweise 40 Punkte für Note 6 nötig, so ist die Notenfunktion

$$n(p) = \begin{cases} 1 & \text{falls } p < 0; \\ 1 + 5 \cdot \frac{p}{40} = \frac{5}{40}p + 1 & \text{falls } 0 \leq p \leq 40; \\ 6 & \text{falls } p > 40 \end{cases}$$